Sets
(Rosen, Chapter 2.1 and 2.2)

TOPICS
- Discrete math
- Set Definition
- Set Operations
- Tuples

What is ‘discrete’?
- Consisting of distinct or unconnected elements, not continuous (calculus)
- Helps us in Computer Science
  - How many values can be represented by an integer primitive type?
  - How many values can be represented using a floating-point format?
  - How many valid Internet address are there?
  - How many steps are required to sort a list?
  - How can we prove our algorithm is more efficient?
  - What is the probability of winning the lottery?
  - How many ways are there to choose a valid password on our computer system?

Why Study Discrete Math?
- Digital computers are based on discrete units of data (bits).
- Therefore, both a computer’s structure (circuits) and operations (execution of algorithms) can be described by discrete math
- A generally useful tool for rational thought! Prove your arguments.

Uses for Discrete Math in Computer Science
- Advanced algorithms and data structures
- Programming language compilers and interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics and animation algorithms, game engines, …
- i.e., almost the entire field of computer science!
What is a set?

- \{1, 2, 3\} is the set containing "1" and "2" and "3."
- An unordered collection of objects
  - \{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
  - \{a, b, c\} = \{b, c, a\} = \{c, b, a\} = \{a, c, b\}
  - \{()\} \// empty set

Finite Sets (limited number of elements)
- \(V = \{a, e, i, o, u\}\) is the Set of vowels
- \(O = \{1, 3, 5, 7, 9\}\) is the Set of odd numbers < 10
- \(F = \{a, 2, Fred, New\ Jersey\}\)
- Boolean data type used frequently in programming
  - \(B = \{0, 1\}\)
  - \(B = \{false, true\}\)
- Seasons = \{spring, summer, fall, winter\}
- ClassLevel = \{Freshman, Sophomore, Junior, Senior\}

Infinite Sets (without end, unending)
- \(\mathbb{N} = \{0, 1, 2, 3, \ldots\}\) is the Set of natural numbers
- \(\mathbb{Z} = \{-\ldots, -2, -1, 0, 1, 2, \ldots\}\) is the Set of integers
- \(\mathbb{Z}^+ = \{1, 2, 3, \ldots\}\) is the Set of positive integers
- \(\mathbb{R}\) is the Set of real numbers
- \(\mathbb{R}^+\) is the Set of positive real numbers
- \(\mathbb{C}\) is the Set of complex numbers

Which of these can be represented on a computer?
What is a set?

- Infinite vs. finite
  - If finite, then the number of elements is called the \textit{cardinality}, denoted $|S|$.
  - $V = \{a, e, i, o, u\} \quad |V| = 5$
  - $F = \{1, 2, 3\} \quad |F| = 3$
  - $B = \{0,1\} \quad |B| = 2$
  - $S = \{\text{spring, summer, fall, winter}\}$

Example sets

- Alphabet
  - All characters
- Booleans: true, false
- Numbers:
  - $\mathbb{N} = \{0,1,2,3,\ldots\}$ Natural numbers
  - $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ Integers
  - $\mathbb{R}$ Real Numbers
  - $\mathbb{Q} = \{ p / q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \}$ Rational
- Note that:
  - $\mathbb{Q}$ and $\mathbb{R}$ are not the same. $\mathbb{Q}$ is a subset of $\mathbb{R}$.
  - $\mathbb{N}$ is a subset of $\mathbb{Z}$.

Venn Diagram

- Graphical representation of set relations

What is a set?

- Defining a set:
  - List the members
  - Use a set builder: $x$ such that $x$ holds a certain characteristic or $\{ x \mid \text{characteristic A} \}$
- Examples
  - $\{ x \mid x \text{ is prime } \}$ is read as set of $x$ such that $x$ is prime
  - $\{ x \mid x \text{ is odd } \}$ is read as set of $x$ such that $x$ is odd
  - $O = \{ x \in \mathbb{N} \mid x \text{ is odd and } x < 10000 \}$
Equality

- Two sets are **equal if and only if (iff)** they have the same elements.
- We write \( A = B \) when for all elements \( x \), \( x \) is a member of the set \( A \) iff \( x \) is also a member of \( B \).
  - Notation \( \forall x \{ x \in A \iff x \in B \} \)
  - For all values of \( x \), \( x \) is an element of \( A \) if and only if \( x \) is an element of \( B \)

Set operations

- \( A = \{a, b, c, d\} \)
- \( B = \{b, c, d, e\} \)
- Union: \( A \cup B = \{a, b, c, d, e\} \)
- Intersection: \( A \cap B = \{b, c, d\} \)
- Set difference: \( A - B = \{a\} \)
- Set difference: \( B - A = \{e\} \)

Operation 1: Union

- The union of sets \( A \) and \( B \) is the set containing those elements in \( A \) or \( B \).
- Notation: \( A \cup B \)
- Example: \( \{1, 2, 3\} \) union \( \{1, 3, 5\} \) is?
  - **Answer:** \( \{1, 2, 3, 5\} \)

Operation 2: Intersection

- The intersection of sets \( A \) and \( B \) is the set containing those elements in both \( A \) and \( B \).
- Notation: \( A \cap B \)
- Example: \( \{1, 2, 3\} \) intersection \( \{1, 3, 5\} \) is?
  - The sets are disjoint if their intersection produces the empty set.
  - **Answer:** \( \{1, 3\} \)
Operation 3: Difference

- The difference of A and B is the set containing those elements that are in A but not in B.
- Notation: \( A - B \)
- Aka the complement of B with respect to A
- Example: \{1, 2, 3\} difference \{1, 3, 5\} is?
- Can you define Difference using union, complement and intersection?
- Answer: \{2\}

Operation 3: Complement

- The complement of set A is the complement of A with respect to \( U \), the universal set.
- Notation: \( \overline{A} \)
- Example: the complement of \{1, 2, 3\} is?
- Depends on the universal set, say positive integers?
- Answer: \{4, 5, 6, ...\}

Identities

<table>
<thead>
<tr>
<th>Identity</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity</td>
<td>( A \cup \emptyset = A, A \cap U = A )</td>
</tr>
<tr>
<td>Commutative</td>
<td>( A \cup B = B \cup A, A \cap B = B \cap A )</td>
</tr>
<tr>
<td>Associative</td>
<td>( A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C )</td>
</tr>
<tr>
<td>Distributive</td>
<td>( A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C) )</td>
</tr>
<tr>
<td>Complement</td>
<td>( A \cup \overline{A} = U, A \cap \overline{A} = \emptyset )</td>
</tr>
</tbody>
</table>

Subset

- The set A is said to be a subset of B iff for all elements \( x \) of A, \( x \) is also an element of B. But not necessarily the reverse...
- Notation: \( A \subseteq B \) \( \forall x \in A \rightarrow x \in B \)  
  - Unidirectional implication
  - \( \{1,2,3\} \subseteq \{1,2,3\} \)
  - \( \{1,2,3\} \subseteq \{1,2,3,4,5\} \)
  - What is the cardinality between sets if \( A \subseteq B \)?
Subset

- Subset is written: $\subseteq$
- **Proper subset** is when $A$ is a subset of $B$, but $B$ is not a subset of $A$. Notation: $\subset$
- $\forall x ((x \in A) \rightarrow (x \in B)) \land \exists x ((x \in B) \land (x \not\in A))$
- All values $x$ in set $A$ also exist in set $B$, but
- There is at least 1 value $x$ in $B$ that does not exist in $A$
- $\{1,2,3\} \subset \{1,2,3,4,5\}$
- $A = \{1,2,3\}$.
- $B = \{1,2,3,4,5\}$
- $A \subset B$ means that $|A| < |B|$.

Empty Set

- **Empty set** has no elements and therefore is the subset of all sets. $\{\}$ Notation: $\emptyset$
- Is $\emptyset \subseteq \{1,2,3\}$?
- Yes! $\forall x (x \in \emptyset) \rightarrow (x \in \{1,2,3\})$ holds, because $(x \in \emptyset)$ is false.
- The cardinality of $\emptyset$ is zero: $|\emptyset| = 0$.
- Consider the set containing just the empty set: $\{\emptyset\}$.
- Yes, this is indeed a set: $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.

Set Theory - Definitions and notation

- **Quiz time:**
  - $A = \{ x \in N \mid x \leq 2000 \} \text{ What is } |A| = 2001$?
  - $B = \{ x \in N \mid x \geq 2000 \} \text{ What is } |B| = \text{ Infinite}!$
  - Is $\{x\} \subseteq \{x\}$? Yes
  - Is $\{x\} \in \{x,(x)\}$? Yes
  - Is $\{x\} \subseteq \{x,(x)\}$? Yes
  - Is $\{x\} \in \{x\}$? No

Powerset

- The powerset of a set is the set containing all the subsets of that set.
- Notation: $P(A)$ is the powerset of set $A$.
- Fact: $|P(A)| = 2^{|A|}$.
  - If $A = \{x, y\}$, then $P(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$
  - If $S = \{a, b, c\}$, what is $P(S)$?
Powerset example

- Number of elements in powerset = $2^n$ where $n =$ # elements in set
- if $S$ is the set {a, b, c} then the powerset of $S$ is:
  - {} (the empty set)
  - {a}
  - {b}
  - {c}
  - {a, b}
  - {a, c}
  - {b, c}
  - {a, b, c}

and hence the power set of $S$ is

{ {}, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c} }  

Useful Example

- Consider binary numbers
  - E.g. 0101
- Let every bit position {1, ..., n} be an item
  - Position $i$ is in the set if bit $i$ is 1
  - Position $i$ is not in the set if bit $i$ is 0
- What is the set of all possible N-bit numbers?
  - The powerset of {0, ..., n-1}!

Cartesian Product

- $A = \{ a, b \}$
- $B = \{ 1, 2, 3 \}$
- $A \times B = \{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$

Venn Diagram

- Graphical representation of set relations
- Universal set is all objects under consideration, represented by $U$
Why sets?

- Programming - Recall a class... it is the set of all its possible objects.
- We can restrict the type of an object, which is the set of values it can hold.
  - Example:
    - Data Type: 
      - int: set of integers
      - char: set of characters
    - Is \( \mathbb{N} \) the same as the set of integers in a computer?

Set Identities

- \( A \cap U = A, A \cup \emptyset = A \)  
  - Identity Laws
- \( A \cup U = U, A \cap \emptyset = \emptyset \)  
  - Domination Laws
- \( A \cup A = A, A \cap A = A \)  
  - Idempotent Laws
- \( A \cup B = B \cup A \)  
  - Commutative Laws
- \( A \cap B = B \cap A \)  
  - Commutative Laws
- \( A \cup \overline{A} = U, A \cap \overline{A} = \emptyset \)  
  - Complement Laws
- ...

De Morgan’s Law

- \( A \cap B = \overline{A} \cup \overline{B} \)
- \( A \cup B = \overline{A} \cap \overline{B} \)
- Proof in textbook, or draw Venn diagram

Order Matters

- What if order matters?
  - Sets disregard ordering of elements
  - If order is important, we use tuples
  - If order matters, then are duplicates important too?
Tuples

- Order matters
- Duplicates matter
- Represented with parens ( )
- Examples
  - \((1, 2, 3) \neq (3, 2, 1) \neq (1, 1, 2, 3, 3)\)

The ordered \(n\)-tuple \((a_1, a_2, ..., a_n)\) is the ordered collection that has \(a_i\) as its first element \(a_2\) as its second element ... and \(a_n\) as its \(n\)th element.

- An ordered pair is a 2-tuple.
- Two ordered pairs \((a, b)\) and \((c, d)\) are equal iff \(a = c\) and \(b = d\) (e.g. NOT if \(a = d\) and \(b = c\)).
- A 3-tuple is a triple; a 5-tuple is a quintuple.

In programming?

- Let’s say you’re working with three integer values, first is the office room # of the employee, another is the # years they’ve worked for the company, and the last is their ID number.
- Given the following set \(\{32, 28, 30\}\), how many years has the employee worked for the company, and the last is their ID number.
- What if the set was \(\{30, 32, 28\}\)?
- Given the 3-tuple \((32, 28, 30)\) can we identify the number of years the employee worked?

Why Tuples?

- Because ordered \(n\)-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position \((2, 3)\) in a maze: `new Mouse(2, 3)`
- Can we reverse the order of the parameters?
- From Java, `Math.pow(1, 2)`
What's next?

- Chapter 1 of Rosen
  - Propositional Logic
  - Propositional Equivalences
  - Predicates
  - Quantifiers
  - Nested Quantifiers
  - Rules of Inference
  - Proofs!