Functions and Sequences
(Rosen, Chapter 2.3-2.4)

TOPICS

• Functions
• Cartesian Products
• Sequences
• Geometric Progressions

Functions (Math)

• Function = mappings or transformations
• Math examples:
  \( f(x) = x \)
  \( f(x) = x + 1 \)
  \( f(x) = 2x \)
  \( f(x) = x^2 \)

Function Definitions

• A function \( f \) from sets \( A \) to \( B \) assigns exactly one element of \( B \) to each element of \( A \).

• Example: the floor function

<table>
<thead>
<tr>
<th>Domain ( A )</th>
<th>Codomain ( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>1</td>
</tr>
<tr>
<td>1.6</td>
<td>2</td>
</tr>
<tr>
<td>5.0</td>
<td>3</td>
</tr>
<tr>
<td>4.8</td>
<td>4</td>
</tr>
<tr>
<td>2.3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

What’s the difference between codomain and range?

Range: \{1,2,4,5\}

Range contains the codomain values that \( A \) maps to

Functions (Computer Sciences)

• In Programming
  – Function header definition example
    \[
    \text{int floor(float real)} \\
    \{ \\
    \} \\
    \]
  • Domain = \( \mathbb{R} \)
  • Codomain = \( \mathbb{Z} \)
Other Functions

- The **identity** function, \( f_{\text{id}} \), on \( A \) is the function where: 
  \[ f_{\text{id}}(x) = x \]
  for all \( x \) in \( A \).
  - \( A = \{a,b,c\} \) and \( f(a) = a, f(b) = b, f(c) = c \)

- The **successor** function, \( f_{\text{succ}}(x) = x + 1 \), on \( Z \), maps a number into the number following it.
  - \( f(1) = 2 \)
  - \( f(-17) = -16 \)
  - \( f(a) \) Does NOT map to \( b \)

- The **predecessor** function, \( f_{\text{pred}}(x) = x - 1 \), on \( Z \), maps a number into the number before it.
  - \( f(1) = 0 \)
  - \( f(-17) = -18 \)

Other Functions

- \( f_{\text{NEG}}(x) = -x \), also on \( R \) (or \( Z \)), maps a value into the negative of itself.

- \( f_{\text{SQ}}(x) = x^2 \), maps a value, \( x \), into its square, \( x^2 \).

- The **ceiling** function: \( \text{ceil}(2.4) = 3 \).

What is NOT a function?

- Consider \( f_{\text{SQRT}}(x) \) from \( Z \) to \( R \).
- This does **not** meet the given definition of a function, because \( f_{\text{SQRT}}(16) = \pm 4 \).
- In other words, \( f_{\text{SQRT}}(x) \) assigns exactly one element of \( Z \) to two elements of \( R \).

Note that the convention is that \( \sqrt{x} \) is always the positive value. 
\[ f_{\text{SQRT}}(x) = \sqrt{x} \]

No Way! Say it ain’t so!!
1 to 1 Functions

- A function $f$ is said to be one-to-one or injective if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.
- Example: the square function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$

1 2 3 4 ...
 1 2 3 4 ...
 9 ...
16

1 to 1 Functions, cont.

- Is square from $\mathbb{Z}$ to $\mathbb{Z}$ an example?
  - NO!
  - Because $f_{SQ}(-2) = 4 = f_{SQ}(+2)$!
- Is floor an example?
  - INCONCEIVABLE!!
- Is identity an example?
  - Unique at last!!

How dare they have the same codomain!

Increasing Functions

- A function $f$ whose domain and co-domain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$ and strictly increasing if $f(x) < f(y)$, whenever
  - $x < y$ and
  - $x$ and $y$ are in the domain of $f$.
- Is floor an example?
  - NO floor is an increasing function
  - But it is NOT STRICTLY increasing.

1.5 < 1.7 and floor(1.5) = 1 = floor(1.7)
and 1.2 < 2.2 and floor(1.2) = 1 = floor(2.2),

- Is square an example?
  - No square is NOT an increasing function UNLESS...

How is Increasing useful?

- Most programs run longer with larger or more complex inputs.
- Consider the maze:
  – Larger maze may (in the worst case) take longer to get out.
  – Maze with more walls may (in the worst case) take longer to get out.
- Consider looking up a telephone number in the paper directory...
**Cartesian Products and Functions**

- A function with multiple arguments maps a Cartesian product of inputs to a codomain.
- Example:
  - `Math.min` maps $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$
    ```java
    int minVal = Math.min(23, 99);
    ```
  - `Math.abs` maps $\mathbb{Q}$ to $\mathbb{Q}^+$
    ```java
    int absVal = Math.abs(-23);
    ```

**Quiz Check**

- Is the following an increasing function?
  - $\mathbb{Z} \to \mathbb{Z}$ $f(x) = x + 5$
  - $\mathbb{Z} \to \mathbb{Z}$ $f(x) = 3x - 1$

**Sequences**

*Your book’s definition:*

A sequence is a function from a subset of the integers (usually $\{1, 2, 3, \ldots\}$) to a set $S$.

In other words:
- it’s an ordered set, where
- a function describes the mapping from position $(1, 2, 3, \ldots)$ to element

**An example sequence**

- $a_n = \frac{1}{n}$
- What happens to $a_n$ when $n$ gets large?
  - $a_n$ gets small, but stays positive
  - $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$
Example Sequences:

- Sequence \( \{a_n\} \) where \( a_n = 1 - n \)
- What is this sequence?
  - 0, -1, -2, -3, ...
- What is the sum of the first 5 terms?
  - \( 0 + (-1) + (-2) + (-3) + (-4) = -11 \)

Important Sequences:

- Arithmetic Progressions:
  - \( S = a, a + d, a + 2d, ..., a + nd, ... \)
  - Example: write a loop that
    1. Reads in a new string from the terminal
    2. Compares it to every previous string, to see if it is new
  - How many comparisons does this program do?

Notation: \( \Sigma \)

- After N iterations, the number of comparisons is:
  \[ 0 + 1 + 2 + ... + (N-1) \]
  This is written as:
  \[ \sum_{i=0}^{n-1} i \]
- In English: the sum of \( i \) from \( i=0 \) to \( i=n-1 \)
- Question: what is the sum from 0 to N-1?
  - Try with N=5, N=10.
  - See if you can arrive at a general formula.

Geometric Progressions

- Geometric Progression:
  - \( S = a, ar, ar^2, ..., ar^n, ... \)
  - Example: how many nodes in a binary tree?
    - Answer: \( 1 + 2 + ... + 2^n + ... \)
  - New question: what is this sum from 1 to N?
- Try with \( n=5 \)