Propositional Logic (Rosen, Chapter 1.1 – 1.3)

TOPICS

• Propositional Logic
• Truth Tables
• Implication
• Logical Proofs

What is logic?

Logic is a truth-preserving system of inference

Truth-preserving: If the initial statements are true, the inferred statements will be true

System: a set of mechanistic transformations, based on syntax alone

Inference: the process of deriving (inferring) new statements from old statements

Propositional Logic

• A proposition is a statement that is either true or false
• Examples:
  • This class is CS160 (true)
  • Today is Sunday (false)
  • It is currently raining in Singapore (???)
• Every proposition is true or false, but its truth value (true or false) may be unknown
Propositional Logic (2)

- A propositional statement is one of:
  - A simple proposition
    - denoted by a capital letter, e.g. 'A'.
  - A negation of a propositional statement
    - e.g. \( \neg A \) : "not A"
  - Two propositional statements joined by a connective
    - e.g. \( A \land B \) : "A and B"
    - e.g. \( A \lor B \) : "A or B"
  - If a connective joins complex statements, parenthesis are added
    - e.g. \( A \land (B \lor C) \)

Truth Tables

- The truth value of a compound propositional statement is determined by its truth table
- Truth tables define the truth value of a connective for every possible truth value of its terms

Logical negation

- Negation of proposition A is \( \neg A \)
- \( A \): It is snowing.
- \( \neg A \): It is not snowing
- \( A \): Newton knew Einstein.
- \( \neg A \): Newton did not know Einstein.
- \( A \): I am not registered for CS160.
- \( \neg A \): I am registered for CS160.

Negation Truth Table

<table>
<thead>
<tr>
<th>( A )</th>
<th>( \neg A )</th>
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</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
Logical and (conjunction)

- Conjunction of A and B is $A \land B$
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - $A \land B$: CS160 teaches logic and Java.

- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi: $A \land \neg B$

Truth Table for Conjunction

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \land B$</th>
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</thead>
<tbody>
<tr>
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</tbody>
</table>

Logical or (disjunction)

- Disjunction of A and B is $A \lor B$
  - A: Today is Friday.
  - B: It is snowing.
  - $A \lor B$: Today is Friday or it is snowing.

- This statement is true if any of the following hold:
  - Today is Friday
  - It is snowing
  - Both
  - Otherwise it is false

Truth Table for Disjunction

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \lor B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
Exclusive Or

- The "or" connective $\lor$ is inclusive: it is true if either or both arguments are true
- There is also an exclusive or $\oplus$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$A \oplus B$</th>
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<tbody>
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<td>T</td>
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</tbody>
</table>

Confusion over Inclusive OR and Exclusive OR

- Restaurants typically let you pick one (either soup or salad, not both) when they say "The entrée comes with a soup or salad".
- Use exclusive OR to write as a logic proposition
- Give two interpretations of the sentence using inclusive OR and exclusive OR:
  - Students who have taken calculus or intro to programming can take this class

Unidirectional implication

- Shown by $A \rightarrow B$
  - A: A programming homework is due.
  - B: It is Tuesday.
- $A \rightarrow B$:
  - If a programming homework is due, then it must be Tuesday.
  - Programming homeworks are due only on Tuesdays.
- Is this the same as:
  - If it is Tuesday, then a programming homework is due.

Unidirectional implication

- $A \rightarrow B$
  - if A then B
  - B when A
  - A implies B
  - B only if A
  - B whenever A
  - B follows from A
  - B if A
- Implication means that one thing follows from another thing
Unidirectional implication and other connectives

- A: You can access free internet at this hotel.
- B: You are attending a conference.
- C: You are staying at this hotel.

You can access free internet at this hotel only if you are attending a conference or you are staying at this hotel.

A → (B ∨ C)

What if we said “if” instead of “only if”?

Unidirectional implication and other connectives (cont’d)

- A: You can register for CS517.
- B: You have taken CS414.
- C: Your instructor gave an override.

You cannot register for CS517 if you haven’t taken CS414 unless your instructor gave an override.

(¬B ∧ ¬C) → ¬A

Equivalent: A → (B ∨ C)

Unidirectional & Bidirectional Implication

- The unidirectional implication connective is →
- The bidirectional implication connective is ↔
- These, too, are defined by truth tables

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A → B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>T</td>
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</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ↔ B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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Bi-conditional

- A: You can drive a car.
- B: You have a driver’s license.

You can drive a car if and only if you have a driver’s license.

A ↔ B

What if we said “if”?

What if we said “only if”?
Truth tables can also be used to determine the truth values of compound statements, such as $(A \lor B) \land (\neg A)$ (fill this as an exercise).

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$\neg A$</th>
<th>$A \lor B$</th>
<th>$(A \lor B) \land (\neg A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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A tautology is a compound proposition that is always true.

A contradiction is a compound proposition that is always false.

A contingency is neither a tautology nor a contradiction.

A compound proposition is satisfiable if there is at least one assignment of truth values to the variables that makes the statement true.

**Examples**

<table>
<thead>
<tr>
<th>A</th>
<th>$\neg A$</th>
<th>$A \lor \neg A$</th>
<th>$A \land \neg A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

Result is always true, no matter what $A$ is. Therefore, it is a tautology.

Result is always false, no matter what $A$ is. Therefore, it is a contradiction.

**Logical Proof**

- Given a set of axioms
  - Statements asserted to be true
- Prove a conclusion
  - Another propositional statement
- In other words:
  - Show that the conclusion is true whenever the axioms are true.
Proof Method #1: Truth Tables

- Write out the truth tables for the axioms and conclusion.
- The conclusion follows from the axioms if (and only if):

\[
\text{The consequent is true for every row in which all the axioms are true}
\]

Proof by Truth Table: Example

- Prove \((A \lor B)\), given \((A \land B)\)

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(A \land B)</th>
<th>(A \lor B)</th>
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<tbody>
<tr>
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Only row for which axiom is true

Conclusion is true, so it follows from the axioms

Logical Equivalence

- Two compound propositions, \(p\) and \(q\), are logically equivalent if \(p \leftrightarrow q\) is a tautology.
- Notation: \(p \equiv q\)
- De Morgan’s Laws:
  - \(\neg (p \land q) = \neg p \lor \neg q\)
  - \(\neg (p \lor q) = \neg p \land \neg q\)
- How so? Let’s build a truth table!

Prove \(\neg (p \land q) \equiv \neg p \lor \neg q\)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(\neg p)</th>
<th>(\neg q)</th>
<th>((p \land q))</th>
<th>(\neg (p \land q))</th>
<th>(\neg p \lor \neg q)</th>
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<td>T</td>
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</tbody>
</table>
Show \( \neg (p \lor q) \equiv \neg p \land \neg q \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \lor q )</th>
<th>( \neg (p \lor q) )</th>
<th>( \neg p \land \neg q )</th>
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<td>T</td>
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Other Equivalences

- Show \( p \to q \equiv \neg p \lor q \)
- Show Distributive Law:
  \[ p \lor (q \land r) = (p \lor q) \land (p \lor r) \]
More Equivalences

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \land T = p$</td>
<td>Identity</td>
</tr>
<tr>
<td>$p \lor F = p$</td>
<td>Identity</td>
</tr>
<tr>
<td>$p \land q = q \land p$</td>
<td>Commutative</td>
</tr>
<tr>
<td>$p \lor q = q \lor p$</td>
<td>Commutative</td>
</tr>
<tr>
<td>$p \lor (p \land q) = p$</td>
<td>Absorption</td>
</tr>
<tr>
<td>$p \land (p \lor q) = p$</td>
<td>Absorption</td>
</tr>
</tbody>
</table>

See Rosen for more.

Equivalences with Conditionals and Biconditionals

- **Conditionals**
  - $p \rightarrow q = \neg p \lor q$
  - $p \rightarrow q = \neg q \rightarrow \neg p$
  - $\neg(p \rightarrow q) = p \land \neg q$

- **Biconditionals**
  - $p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)$
  - $p \leftrightarrow q = \neg p \leftrightarrow \neg q$
  - $\neg(p \leftrightarrow q) = p \leftrightarrow \neg q$

Prove Biconditional Equivalence

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
<th>$\neg(p \leftrightarrow q)$</th>
<th>$p \leftrightarrow \neg q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

Proof By Truth Table

- If the consequent is true in the truth table whenever the axioms are true, it is proved
  - Warning: when the axioms are false, the consequent may be true or false
- Problem: given $n$ propositions, the truth table has $2^n$ rows
  - Proof by truth table quickly becomes infeasible
### Logical Equivalences (Rosen)

<table>
<thead>
<tr>
<th>Logical Equivalences</th>
<th>DeMorgan's Laws</th>
<th>Distributive Laws</th>
</tr>
</thead>
<tbody>
<tr>
<td>Idempotent Laws</td>
<td>¬(p ∨ q) ≡ ¬p ∨ ¬q</td>
<td>p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)</td>
</tr>
<tr>
<td>p ∨ p ≡ p</td>
<td></td>
<td>p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)</td>
</tr>
<tr>
<td>p ∧ p ≡ p</td>
<td></td>
<td>p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)</td>
</tr>
<tr>
<td>Double Negation</td>
<td>¬¬p ≡ p</td>
<td>Absorption Laws</td>
</tr>
<tr>
<td>¬¬p ≡ p</td>
<td></td>
<td>(p ∧ q) ∨ r ≡ p ∨ (q ∨ r)</td>
</tr>
<tr>
<td>Absorption Laws</td>
<td>¬¬p ≡ p</td>
<td>(p ∧ q) ∧ r ≡ p ∨ (q ∨ r)</td>
</tr>
<tr>
<td>Implication Laws</td>
<td>p → q ≡ ¬p ∨ q</td>
<td></td>
</tr>
<tr>
<td>p → q ≡ ¬p ∨ q</td>
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<td></td>
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<tr>
<td>p ∧ q ≡ q ∧ p</td>
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<tr>
<td>Commutative Laws</td>
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</tbody>
</table>

#### Proof Method #2: Rules of Inference

- A **rule of inference** is a proven relation: when the left hand side (LHS) is true, the right hand side (RHS) is also true.

- Therefore, if we can match an axiom to the LHS by substituting propositions, we can assert the (substituted) RHS.

### Transformation via logical equivalences

Prove: ¬(p ∨ ¬(p ∧ q)) ≡ (¬p ∧ ¬q)

1. ¬(p ∨ ¬(p ∧ q))  
   - Given
2. ¬p ∧ ¬(¬p ∧ q)  
   - DeMorgan’s Law
3. ¬p ∧ ¬(¬p ∨ ¬q)  
   - DeMorgan’s Law
4. ¬p ∧ (p ∨ ¬q)  
   - Double Negation
5. (¬p ∧ p) ∨ (¬p ∧ ¬q)  
   - Distributive Law
6. F ∨ (¬p ∧ ¬q)  
   - Negation Law
7. (¬p ∧ ¬q) ∨ F  
   - Commutative Law
8. (¬p ∧ ¬q)  
   - Identity law

### Applying rules of inference

- Example rule: A, A → B \hspace{1cm} B
  - Read as “A and A → B, therefore B”
  - This rule has a name: **modus ponens**

- If you have axioms C, C → D
  - Substitute C for A, D for B
  - Apply modus ponens
  - Conclude D
Inference properties

- Inference rules are truth preserving
  - If the LHS is true, so is the RHS
- Applied to true statements
  - Axioms or statements proved from axioms
- Inference is syntactic
  - Substitute propositions
    - If A replaces C once, it replaces C everywhere
    - If A replaces C, it only replaces C
  - Apply rule

Rules of Inference (Rosen)

- **Modus Ponens**
  - If p, and p implies q, then q
  - Example:
    - p = it is sunny, q = it is hot
    - p → q, it is hot whenever it is sunny
    - "Given the above, if it is sunny, it must be hot."

- **Modus Tollens**
  - If not q and p implies q, then not p
  - Example:
    - p = it is sunny, q = it is hot
    - p → q, it is hot whenever it is sunny
    - "Given the above, if it is not hot, it cannot be sunny."
Hypothetical Syllogism

- If p implies q, and q implies r, then p implies r

Example:
- p = it is sunny, q = it is hot, r = it is dry
- p → q, it is hot when it is sunny
- q → r, it is dry when it is hot
- “Given the above, it must be dry when it is sunny”

Disjunctive Syllogism

- If p or q, and not p, then q

Example:
- p = it is sunny, q = it is hot
- p ∨ q, it is hot or sunny
- “Given the above, if it is sunny, it must be hot or dry”

Resolution

- If p or q, and not p or r, then q or r

Example:
- p = it is sunny, q = it is hot, r = it is dry
- p ∨ q, it is sunny or hot
- ¬p ∨ r, it is not hot or dry
- “Given the above, if it is sunny or hot, but not sunny or dry, it must be hot or dry”
- Not obvious!

Addition

- If p then p or q

Example:
- p = it is sunny, q = it is hot
- p ∨ q, it is hot or sunny
- “Given the above, if it is sunny, it must be hot or sunny”
- Of course!
Simplification

- If p and q, then p
  Example:
  p = it is sunny, q = it is hot
  p \land q, it is hot and sunny
  “Given the above, if it is hot and sunny, it must be hot”
  Of course!

Conjunction

- If p and q, then p and q
  Example:
  p = it is sunny, q = it is hot
  p \land q, it is hot and sunny
  “Given the above, if it is sunny and it is sunny, it must be hot and sunny”
  Of course!

A Simple Proof

- Given: X, X \rightarrow Y, Y \rightarrow Z, \neg Z \lor W
- Prove: W

  1. X \rightarrow Y  
     - Given
  2. Y \rightarrow Z  
     - Given
  3. X \rightarrow Z  
     - Hypothetical Syllogism (1, 2)
  4. X  
     - Given
  5. Z  
     - Given
  6. \neg Z \lor W  
     - Disjunctive Syllogism (5, 6)
  7. W  

A Simple Proof From Words

“In order to take CS161, I must first take CS160 and either M155 or M160. I have not taken M155 but I have taken CS161. Prove that I have taken M160.”

First step: assign propositions
- A : take CS161
- B : take CS160
- C : take M155
- D : take M160
Now set up the proof

- Axioms:
  - $A \rightarrow B \land (C \lor D)$
  - $A$
  - $\neg C$

- Conclusion:
  - $D$

Now do the proof

1. $A$
   - Given
2. $A \rightarrow B \land (C \lor D)$
   - Given
3. $B \land (C \lor D)$
   - Modus Ponens (1, 2)
4. $(C \lor D)$
   - Simplification
5. $\neg C$
   - Given
6. $D$
   - Disjunctive Syllogism (4, 5)