Counting

Rosen, Chapter 5.1, 5.2, 5.3
Walls and Mirrors, Chapter 3

Spock's dilemma (Walls and mirrors)

- n planets in the solar system
- can only visit k \( \leq n \) of them
- how many options does Spock have?

call it \( C(n,k) \)
C(n,k) choose k out of n

Let's formulate a solution recursively

- consider planet n
  - either visit n  →  Spock must choose k-1 out of n-1
  - or  don't  →  Spock must choose k out of n-1
- Can he always choose not to visit n?
  - No! if n==k he must visit all  →  1 way to choose
- What if there are 0 planets (left) to visit?
  - 1 way: visit none of them

Recurrence relation

- We can write this in a recurrence relation
  - a recursive mathematical expression

\[
C(n,k) = C(n-1,k-1) + C(n-1,k)
\]

- pick n  or  don't

- C(k,k) = 1
- C(n,0) = 1

and we can code this easily as a recursive method.
Combinations

- Spock just counted the number of choices
- How can we enumerate them all?
  How many combinations (subsets) of $k$ out of 0 – (n-1) can we create?

Notice that this is a generalization of the subset problem from Assign 1

Combinations(5,3)

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- what is the largest digit we can place in the first position? Can you generalize that for $C(n,k)$?
How do we do it?

- place digits $d$ from $lo$ to $hi$ in position $p$
- then recursively place digits $d+1$ to $hi$ in position $p+1$
- hi: $n-k$ for pos 0, $n-k+1$ for pos 1, $n-k+p$ for pos $p$
- let's play with the code

Counting: the product rule

- If there are $n_1$ ways of doing one task, and for each of way of doing the first task there are $n_2$ ways of doing a second task, then there are $n_1n_2$ ways of performing both tasks.

Example:
- You have 6 pairs of pants and 10 shirts. How many different outfits does this give?
Relation to Cartesian products

- The **Cartesian product** of sets A and B is denoted by $A \times B$ and is defined as:
  
  $A \times B = \{ (a,b) | a \in A \text{ and } b \in B \}$

- $|A \times B| = |A| \times |B|$

Product rule

- Colorado assigns license plates numbers as a-b-c x-y-z, where a,b,c are digits and x,y,z are letters. How many license plates numbers are possible?
More examples

- How many binary numbers with 7 digits are there?
- How many functions are there from a set with m elements to a set with n elements?
- How many one-to-one functions are there from a set with m elements to a set with n elements?

More examples

- Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$
- Product rule and Cartesian products
  - $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \times |A_2| \times \ldots \times |A_n|$
Sum Rule

- Person X has decided to shop at one store today, either in the north end of town or the south end of town. If X visits the north end, X will shop at one of three stores. If X visits the south end of town then X will shop at one of two stores. How many ways could X end up shopping?

The Sum Rule

- If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, and none of the $n_1$ ways is the same as the $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.
- This is a statement about set theory: if two sets A and B are disjoint then

$$|A \cup B| = |A| + |B|$$
Example

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 4 possible projects. No project is on more than one list. How many possible projects are there to choose from?

The inclusion exclusion principle

- In set theory terms:

  \[ |A \cup B| = |A| + |B| - |A \cap B| \]

- Think of it as Venn Diagrams
Try using inclusion / exclusion

- Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit, and each password must contain at least one digit. How many possible passwords are there?

The inclusion exclusion principle

- How many bit strings of length eight start with a 1 or end with 00?

  1 - - - - - -  how many?
  - - - - - - 0 0 how many?

  if I add these
  how many have I now counted twice?
The pigeonhole principle

- If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more objects.

Examples

- In a group of 367 people, there must be at least two with the same birthday

- A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.
  - How many socks must he take out to be sure that he has at least two socks of the same color?
  - How many socks must he take out to be sure that he has at least two black socks?
Examples

- Show that if five different digits between 1 and 8 are selected, there must be at least one pair of these with a sum equal to 9.

- ask yourself: what are the pigeon holes?
  what are the pigeons?