Motivation

- Suppose we want to prove the predicate $P(n)$: for every positive value of $n$:
  
  \[ 1 + 2 + \ldots + n = \frac{n(n + 1)}{2}. \]

- We observe $P(1), P(2), P(3), P(4)$. Conjecture: $\forall n \in \mathbb{N}, P(n)$.
- Mathematical induction is a proof technique for proving statements of the form $\forall n \in \mathbb{N}, P(n)$.
Proving $P(3)$

- Suppose we know: $P(1) \land \forall n \geq 1, P(n) \to P(n + 1)$. 
  Prove: $P(3)$
- Proof:
  1. $P(1)$. [premise]
  2. $P(1) \to P(2)$. [specialization of premise]
  3. $P(2)$. [step 1, 2, & modus ponens]
  4. $P(2) \to P(3)$. [specialization of premise]
  5. $P(3)$. [step 3, 4, & modus ponens]

We can construct a proof for every finite value of $n$
- Modus ponens: if $p$ and $p \to q$ then $q$

Example: $1 + 2 + \ldots + n = n(n + 1)/2$.

- Verify: $F(1): 1(1 + 1)/2 = 1$.
- Assume: $F(n) = n(n + 1)/2$
- Show: $F(n + 1) = (n + 1)(n + 2)/2$.
  $F(n + 1) = 1 + 2 + \ldots + n + (n + 1)$
  $= F(n) + n + 1$
  $= n(n + 1)/2 + n + 1$ [Induction hyp.]
  $= n(n + 1)/2 + 2(n + 1)/2$
  $= (n + 1)(n + 2)/2$. 

A Geometrical interpretation

1:

2:

3:

Put these blocks, which represent numbers, together to form sums:

\[ 1 + 2 = \]

\[ 1 + 2 + 3 = \]

Area is \( \frac{n^2}{2} + \frac{n}{2} = \frac{n(n + 1)}{2} \)
The Principle of Mathematical Induction

- Let $P(n)$ be a statement that, for each natural number $n$, is either true or false.
- To prove that $\forall n \in \mathbb{N}, P(n)$, it suffices to prove:
  - $P(1)$ is true. (base case)
  - $\forall n \in \mathbb{N}, P(n) \rightarrow P(n + 1)$. (inductive step)
- This is not magic.
- It is a recipe for constructing a proof for an arbitrary $n \in \mathbb{N}$.

Mathematical Induction and the Domino Principle

If the 1st domino falls over and the $n$th domino falls over implies that domino $(n + 1)$ falls over then domino $n$ falls over for all $n \in \mathbb{N}$.

Proof by induction

- 3 steps:
  - Prove P(1). [the basis]
  - Assume P(n) [the induction hypothesis]
  - Using P(n) prove P(n + 1) [the inductive step]

Example

- Show that any postage of \( \geq 8\) can be obtained using 3\(\) and 5\(\) stamps.
- First check for a few values:
  - \(8\) = 3\(\) + 5\(\)
  - \(9\) = 3\(\) + 3\(\) + 3\(\)
  - \(10\) = 5\(\) + 5\(\)
  - \(11\) = 5\(\) + 3\(\) + 3\(\)
  - \(12\) = 3\(\) + 3\(\) + 3\(\) + 3\(\)
- How to generalize this?
Example

- Let $P(n)$ be the sentence “$n$ cents postage can be obtained using 3¢ and 5¢ stamps”.
- Want to show that “$P(k)$ is true” implies “$P(k+1)$ is true” for all $k \geq 8$.

2 cases:
1) $P(k)$ is true and the $k$ cents contain at least one 5¢.
2) $P(k)$ is true and the $k$ cents do not contain any 5¢.

Example

**Case 1:** $k$ cents contain at least one 5¢ coin.

- Replace 5¢ stamp by two 3¢ stamps.
- Then there are at least three 3¢ coins.

**Case 2:** $k$ cents do not contain any 5¢ coin.

- Replace three 3¢ stamps by two 5¢ stamps.
Examples

- Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

- Show that for $n \geq 4$ $2^n < n!$

- Show that $n^3 - n$ is divisible by 3 for every positive $n$.

- Show that $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$

All horses have the same color

- **Base case:** If there is only one horse, there is only one color.

- **Induction step:** Assume as induction hypothesis that within any set of $n$ horses, there is only one color. Now look at any set of $n + 1$ horses. Number them: 1, 2, 3, ..., $n$, $n + 1$. Consider the sets {1, 2, 3, ..., $n$} and {2, 3, 4, ..., $n + 1$}. Each is a set of only $n$ horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n + 1$ horses.

- **What's wrong here?**
Strong induction

- Induction:
  - P(1) is true.
  - ∀n ∈ N, P(n) → P(n + 1).
  - Implies ∀n ∈ N, P(n)

- Strong induction:
  - P(1) is true.
  - ∀n ∈ N, (P(1) ∧ P(2) ∧ ... ∧ P(n)) → P(n + 1).
  - Implies ∀n ∈ N, P(n)

Example

- Prove that all natural numbers ≥ 2 can be represented as a product of primes.

- **Basis:** 2: 2 is a prime.

- **Assume** that 1, 2,…, n can be represented as a product of primes.
Example

- **Show** that \( n+1 \) can be represented as a product of primes.
  - Case \( n+1 \) is a prime: It can be represented as a product of 1 prime, itself.
  - Case \( n+1 \) is composite: Then, \( n + 1 = ab \), for some \( a, b < n + 1 \).
    - Therefore, \( a = p_1p_2 \ldots p_k \) & \( b = q_1q_2 \ldots q_l \), where the \( p_i \)s & \( q_i \)s are primes.
    - Represent \( n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l \).

More examples

- Back to the postage stamp problem. If I know that 8, 9, 10, 11, and 12 can be paid with 3s and 5s, what can I say about 13, 14, ... using strong induction?

- A full binary tree is a binary tree where every internal node has 2 children. Use strong induction to prove that in every full binary tree has one more leaf than internal nodes.
Induction and Recursion

- Induction is useful for proving correctness/design of recursive algorithms
- Example

```java
// Returns base ^ exponent.
// Precondition: exponent >= 0
public static int pow(int x, int n) {
    if (n == 0) {
        // base case; any number to 0th power is 1
        return 1;
    } else {
        // recursive case: x^n = x * x^(n-1)
        return x * pow(x, n-1);
    }
}
```

Induction and Recursion

- \( n! \) of some integer \( n \) can be characterized as:
  \[
  n! = \begin{cases} 
  1 & \text{for } n = 0; \\
  n (n - 1) (n - 2) \ldots 1 & \text{otherwise}
  \end{cases}
  \]
- Want to write a recursive method for computing it. We notice that \( n! = n (n - 1)! \)
- This is all we need to put together the method:

```java
public static int factorial(int n) {
    if (n == 0) {
        return 1;
    } else {
        return n * factorial(n-1);
    }
}
```