Axis Angle Rotation
& Intro to Cameras

Tuesday September 23rd 2014

Axis-Angle Form

\[ R = \langle \theta, W \rangle = \begin{bmatrix} \theta \end{bmatrix} \begin{bmatrix} w_x & w_y & w_z \end{bmatrix} \]

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]

Pros, Cons and going from axis angle to 3x3 matrix form.
Rotation – Matrix ↔ Axis Angle

- In $P' = MP$, the points in $P$ are projected onto the rows of $M$.
- In a rotation matrix:
  - The rows are unit length
    - Otherwise it scales the data...
  - The rows are orthogonal
    - Otherwise it shears the data...

To specify a rotation matrix, just specify the (orthogonal, unit) basis vectors of the new coordinate system!

Axis Angle Rotation

- In the special case that the axis is the Z axis, no problem:

$$\begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 & 0 \\
\sin(\theta) & \cos(\theta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

- Where $\theta$ is the magnitude of the rotation

- But what about all the other possible axes?
Axis Angle (II)

- General rule: if there is a special-case coordinate system that makes life easy, adopt that coordinate system!

- In this case:
  1. Rotate data to make Z the angle of rotation
  2. Rotate about Z
  3. Apply the inverse of the original rotation

Axis Angle (III)

- How do we rotate the data to make the angle of rotation Z?
  - Multiplication is projection onto the rows of M
  - If M is orthonormal, it is a rotation matrix
    - Magnitude of every row is 1
    - Dot product of every pair of rows is 0
  - If the third row is the axis of rotation, Z becomes the axis of rotation!
### Axis Angle (IV)

- **Step 1:** normalize the axis of rotation
  - Write the normalized axis as \( w = (w_x, w_y, w_z, 1) \)
- **Step 2:** pick any axis \( M \) not parallel to \( W \)
  - Heuristic: pick the smallest term in \( w \), set it to 1 and renormalize to create \( m \)
- **Step 3:** create \( U = W \times M \)
- **Step 4:** pick an axis \( v \) perpendicular to \( w \) & \( u \)
  - \( V = W \times U \) (or \( U \times W \))

### Axis Angle (V)

- Now put those together in a rotation matrix:

\[
R_\omega = \begin{bmatrix}
u_x & u_y & u_z & 0 \\
v_x & v_y & v_z & 0 \\
w_x & w_y & w_z & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Axis Angle (VI)

To rotate by $\theta$ around $\omega$:

$$P' = \begin{pmatrix} R^{-1}_\omega & R_{Z\theta} & R_\omega \end{pmatrix} P$$

- $R_\omega$ is from the last slide
- $R_{Z\theta}$ is the rotation matrix about Z, by amount $\theta$
- What about $R^{-1}$?

Axis Angle (V)

Useful math fact: the inverse of an orthonormal matrix is its transpose

$$P' = \begin{pmatrix} R^T_\omega & R_{Z\theta} & R_\omega \end{pmatrix} P$$

That’s how you implement axis-angle rotation!
3D Viewing as Virtual Camera

To take a picture with a camera, or to render an image with computer graphics, we need to:

- Position the camera/viewpoint in 3D space
- Orient the camera/viewpoint in 3D space
- Focus camera -- *we won’t do this step*
- Crop image to the aperture/window

Perspective ...
Orthographic Projection

If not for the fog, you could see forever ... and nothing ever would look smaller.

Orthographic / Perspective
Think About Rays
Perspective Always Better?

No! Technical programs, including for example Maple, often favor orthographic projection.

Perspective Projection

- Light rays pass through the focal point.
  - a.k.a. principal reference point PRP.
- The image plane is an infinite plane in front of (or behind) the focal point.
- Images are formed by rays of light passing through the image plane
- Common convention:
  - Image points are \((u, v)\)
  - World points are \((x, y, z)\)
A Tad Bit Of History

- The Camera Obscura - see Wikipedia

- Pre-dates photographic cameras.
  - Theory: Mo-Ti (China, 470-390 BC)
  - Practice: Abu Ali Al-Hasan Ibn al-Haitham (~1000 AD)

Why “Pinhole” Camera?

- Because you can build a camera that exactly fits this description:
  - Create a fully-enclosed black box
    - So that no light enters
  - Put a piece of film inside it, facing front
  - Punch a pin-hole in the front face of the box

- What is this camera’s depth-of-field?
- What doesn’t this camera have?
- Why don’t we build cameras this way?
Now Math - Orthographic

- Simply drop a dimension.
  $\begin{bmatrix} u \\ v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

- Think of a bug hitting a windshield.
- No more z axis!
  (no more bug)

Photo by Brian, Jeff Booth site
www.jeffbooth.net (Creative Common License)