Probabilistic Methods: Overview

- Concrete numbers in presence of uncertainty
- Probability
  - Disjoint events
  - Statistical dependence
- Random variables and distributions
  - Discrete distributions: Binomial, Poisson
  - Continuous distributions: Gaussian, Exponential, Weibull
- Stochastic processes
  - Markov
  - Poisson
Basics

- Probability of an event $A$ (between 0 and 1)
  \[ P(A) = \frac{n}{N} \]
  if $A$ occurs $n$ times among $N$ equally likely outcomes.
- Ex: Roll of a die
  \[ P(\text{odd}) = \frac{3}{6} = 0.5 \]
- If more information is available, probability of the same event changes. If we know die is loaded, perhaps
  \[ P(\text{odd}) = 0.6 \]

Basics (2)

- Union:
  \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
- Ex: Roll of a die
  \[ P(\text{outcome even} \cup \text{outcome} \leq 3) = P(\text{even}) + P(\leq 3) - P(\text{even} \cap \leq 3) = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6} \]
- If $A$ and $B$ are disjoint, i.e. if $A \cap B = \emptyset$,
  \[ P(A \cup B) = P(A) + P(B) \]
- \[ P(\overline{A}) = 1 - P(A) \]
Basics (3)

- **Conditional probability**
  \[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ for } P(B) > 0 \]

- **If A and B are independent**, \( P(A \mid B) = P(A) \). Then \( P(A \cap B) = P(A) \cdot P(B) \).

- **If A can be divided into disjoint \( A_i \), \( i=1,\ldots,n \), then**
  \[ P(B) = \sum_i P(B \mid A_i) P(A_i) \]

Random Variables

- **X**: random variable (ex: height of a randomly chosen student)
- **x**: one specific value (say 5’9”)

<table>
<thead>
<tr>
<th></th>
<th>continuous</th>
<th>discrete</th>
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<tbody>
<tr>
<td>( f(x)dx )</td>
<td>( P{x \leq X \leq x + dx} )</td>
<td>( p(x_i) )</td>
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<tr>
<td>( F(x) )</td>
<td>( \int_{x_{\text{min}}}^{x} f(x)dx )</td>
<td>( \sum_{i=\text{min}}^{i=\text{max}} p(x_i) )</td>
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<td>( E(X) )</td>
<td>( \int_{x_{\text{min}}}^{x_{\text{max}}} x f(x)dx )</td>
<td>( \sum_{i=\text{min}}^{i=\text{max}} x_i p(x_i) )</td>
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Distributions, Binomial

- Note that \( \int_{x_{\min}}^{x_{\max}} f(x) \, dx = 1 \) \( \sum_{i_{\min}}^{i_{\max}} p(x_i) = 1 \)

- Major distributions:
  - Discrete: Binomial, Poisson
  - Continuous: Gaussian, exponential

- **Binomial** (success/failure)
  - Prob. of \( r \) occurrences in \( n \) trials, prob. of one success being \( p \)

\[
f(r) = \binom{n}{r} p^r (1-p)^{n-r} \quad \text{for} \quad r = 0, \ldots, n
\]

Incidentally \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

Distributions: Poisson

- Poisson: also discrete. \( \lambda = \) occurrence rate of something.
  - Probability of \( r \) occurrences in time \( t \) is

\[
f(r) = \frac{e^{-\lambda t} (\lambda t)^r}{r!} \quad \text{[also sometimes} \ f(x) = \frac{\lambda e^{-\lambda}}{x!} ]
\]

YKM
Distributions: Gaussian

- Continuous. Also termed Normal (Laplacian in France)

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \]

\(-\infty \leq x \leq +\infty\)

\(\sigma\): standard deviation

\(\mu\): mean

Bell-shaped curve

Normal distribution (2)

- Tables for normal distribution are available, often in terms of standardized variable \(z=(x-\mu)/\sigma\).
- \((\mu-\sigma, \mu+\sigma)\) includes 68.3\% of the area under the curve.
- \((\mu-3\sigma, \mu+3\sigma)\) includes 99.7\% of the area under the curve.
- Central Limit Theorem: Sum of a large number of independent random variables tends to have a normal distribution.
Exponential & Weibull Dist.

- **Exponential**: Continuous. $\lambda$: exit or failure rate.
  - $\Pr\{\text{exit the good state during} \ (t, \ t+dt)\} = e^{-\lambda t} \ dt$
  - Density function
    
    $$f(t) = \lambda e^{-\lambda t} \quad 0 < t \leq \infty$$
    
    - Prob. of no exit in $(0,t)$ is $1-F(t)$. (Reliability)
  - **Weibull Distribution**: 2-parameter generalization of exponential. Better fit in many cases. More later.

Variance & Covariance

- **Variance**: a measure of spread
  - $\text{Var}(X) = \mathbb{E}[(X-\mu)^2]$
  - Standard deviation $= (\text{Var}(x))^{1/2}$
  - $\sigma$ = standard deviation (usually for normal dist)
  - **Covariance**: a measure of statistical dependence
    - $\text{Cov}(X,Y) = \mathbb{E}[(X-\mu_x)(Y-\mu_y)]$
    - Correlation coefficient: normalized
      
      $$\rho_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y}$$
      
      Note that $0 < |\rho_{xy}| < 1$
Stochastic Processes

- **Stochastic process**: that takes random values at different times.
  - Can be continuous time or discrete time
- **Markov process**: discrete-state, continuous time. Transition probability from state i to state j depends only on state i (memory-less)
- **Markov chain**: discrete-state, discrete time
- **Poisson process**: A Markov counting process \(N(t), t \geq 0\), \(N(t)\) is the number of arrivals up to time \(t\).

Poisson Process: definition

- **Poisson process**: A Markov counting process \(N(t), t \geq 0\), \(N(t)\) is the number of arrivals up to time \(t\).
  - \(N(0)=0\)
  - \(P(\text{an arrival in time } \Delta t)=\lambda \Delta t\)
  - No simultaneous arrivals
Poisson process: analysis

\[ P_0 = P\{\text{process in state 0}\} \]
\[ P_0(t + \Delta t) = P_0(t)[1 - \lambda \Delta t] \]
\[ \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) \]
\[ \frac{dP_0(t)}{dt} = -\lambda P_0(t) \]

Solution:
\[ \ln(P_0(t)) = -\lambda t + C \]
\[ P_0(t) = C_2 e^{-\lambda t} \]
Since \( P_0(0) = 1, C_2 = 1, \)
\[ P_0(t) = e^{-\lambda t} \]

Poisson Process: more

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In general,
\[ \frac{dP_0(t)}{dt} = -\lambda P_0(t) + \lambda P_{i-1}(t) \]
Solving recursively,
\[ P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad n = 0,1,\ldots \]
Poisson Process: Time between Two Events

\[ P(T > t) = P\{\text{no arrival in } (0, t)\} = e^{-\lambda t} \]

Thus

\[ F(t) = P\{0 \leq T \leq t\} = 1 - e^{-\lambda t} \]

\[ f(t) = \lambda e^{-\lambda t} \quad \text{Exponential distribution} \]