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Fault Tolerant Ring Embedding in Tetravalent Cayley Network Graphs*

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Abstract

Our purpose in the present paper is to investigate the structural properties of the newly proposed Cayley networks of constant node degree 4. We show that the graph contains rings of maximal length in presence of multiple faults. Our results provide further evidence to the usefulness and robustness of these network graphs.

1 Introduction

Cayley graphs have drawn considerable interest in the recent past for designing interconnection networks because of many desirable properties like low diameter, low degree, high fault tolerance etc. Cayley graphs are based on permutation groups and include a large number of families of graphs, like star graphs [AK89, AK87], hypercubes [BA84], pancake graphs [AK89, QAM94] and others [Sch91, DT92]. All Cayley graphs are regular, but almost none of the Cayley graphs studied so far offer constant node degree (where node degree does not change with size or dimension of the network). There are a number of applications where we need such constant degree networks; in VLSI design we need

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them for area efficient layout [CAB93]; there are applications where the computing nodes can have only a fixed number of I/O ports[SP89]. Constant degree network graphs are of considerable practical importance since De Bruijn graphs are being used for designing a 8096 node multiprocessor at JPL for the Galileo project [Pra91]. There exist graphs in the literature with bounded node degree; most popular among them are De Bruijn graphs [PR82], Moebius graphs [LS82], and Cube-Connected Cycles [PV81]. Most of these graphs, except Cube-Connected Cycles, are not regular and they offer low vertex connectivity (fault tolerance); for example, almost all nodes in a De Bruijn graph have a node degree of 4 while the vertex connectivity of the network is only 2. Recently authors in [VS96] have developed a new family of Cayley graphs of constant degree 4 where they have shown that the graph is regular, has a logarithmic diameter and has a vertex connectivity 4 (thus, maximally fault tolerant); an optimal routing algorithm has also been developed. It is to be noted that these graphs seem to be similar to butterfly network with wraparound [ABR90]. Note that Cube-Connected Cycles are also regular Cayley graphs, but the graphs TCN_n in [VS96] have a higher vertex connectivity (hence higher fault tolerance) and it accommodates a larger number of nodes than cube-connected cycle graph for the same diameter.

Our purpose in the present paper is to further investigate the topological properties of these tetravalent Cayley networks (we call TCN_n). Specifically, it is important to be able to simulate cycles of different lengths. Embedding of rings of maximal lengths is essential to run parallel algorithms developed for arrays and vectors. The presence of a Hamiltonian in TCN_n shown in [VS96, Sto87]. An important related question is "is a cycle of length $N - c$, where N is the number of nodes in the graph and c is a *constant*, contained in the graph, in the presence of a single arbitrary faulty node". The question is answered affirmative for hypercubes [CL91]; we do not know of any constant node degree graph with that property. We develop structural properties of TCN_n, enumerate cycles of different lengths in TCN_n, show that TCN_n always has a Hamiltonian and then show that the graph TCN_n (where $N = n \times 2^n$) does contain a cycle of length $N - 2$ in presence a single arbitrary node failure and a cycle of length at least $N - 4$ in presence of two arbitrary node failures. Thus, we show that the graph TCN_n is not only an attractive alternative to De Bruijn graphs for VLSI implementation in terms of regularity and greater fault tolerance without additional cost, but also is competitive with hypercubes in terms of embedding fault tolerant rings of maximal length.

2 Tetravalent Cayley Networks TCN_n

Tetravalent Cayley Networks TCN_n is defined as a graph on $n \times 2^n$ vertices for any integer n, $n \geq 3$; each vertex is represented by a circular permutation of n symbols in lexicographic order where each symbol may be present in either uncomplemented or complemented form. Let t_k, $1 \leq k \leq n$ denote the k-th symbol in the set of n symbols (we use English alphabets as symbols; thus for $n = 4$, $t_1 = a$, $t_2 = b$, $t_3 = c$ and $t_4 = d$). We use t_k^* to denote either t_k or $\bar{t}_k$. Thus, for n distinct symbols, there are exactly n different cyclic permutation of the symbols in lexicographic order and since each symbol can be present in either complemented or uncomplemented form, the vertex set of TCN_n (i.e. the underlying group Γ) has a cardinality of $n.2^n$ (for example, for $n = 3$, the number of vertices in G_3 is 24; abc, cab, $\bar{c}ab$ are valid nodes while acb or bac are not). Let I denote the *identity permutation* $t_1 t_2 \cdots t_n$. Since each node is some cyclic permutation of the n symbols in lexicographic order, then if $a_1 a_2 \cdots a_n$ denotes the label of an arbitrary node and $a_1 = t_k^*$ for some integer k, then for all i, $2 \leq i \leq n$, we have $a_i = t^*_{(k+i) \bmod n+1}$. The edges of TCN_n are defined by the following four generators in the graph:

$$g(a_1 a_2 \cdots a_n) = a_2 a_3 \cdots a_n a_1$$
$$f(a_1 a_2 \cdots a_n) = a_2 a_3 \cdots a_n \bar{a}_1$$
$$g^{-1}(a_1 a_2 \cdots a_n) = a_n a_1 \cdots a_{n-1}$$
$$f^{-1}(a_1 a_2 \cdots a_n) = \bar{a}_n a_1 \cdots a_{n-1}$$

Remark 1 *Figure 1 shows the proposed degree four Cayley graph TCN_3 of dimension 3. The set of four generators, $\Omega = \{f, g, f^{-1}, g^{-1}\}$ closed under inverse; in particular g is inverse of g^{-1} and f is inverse of f^{-1}; thus the edges in TCN_n are bidirectional.*

This graph TCN_n, $n \geq 3$ is regular with node degree 4, has a diameter $\lfloor \frac{3n}{2} \rfloor$ (logarithmic in number of nodes) and is maximally fault tolerant (vertex connectivity is 4); see [VS96] for the details as well as an optimal routing algorithm.

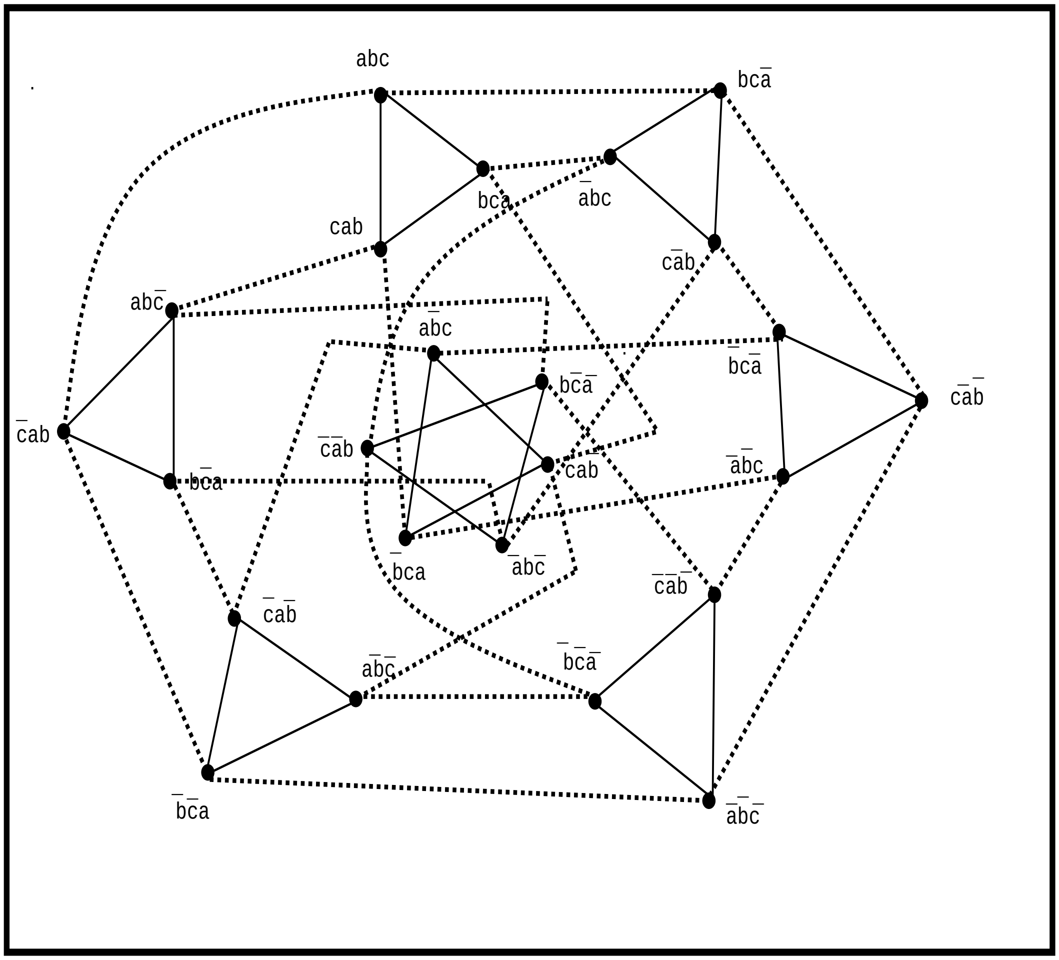

Figure 1: Example Graph for $n = 3$ (24 nodes)

3 Fault Tolerant Ring Embedding in TCN_n

Since TCN_n is a Cayley graph, it is vertex symmetric [AK89], i.e., we can always view the distance between any two arbitrary nodes as the distance between the source node and the identity permutation by suitably renaming the symbols representing the permutations. Thus, in our subsequent discussion about the distance between a source node and a destination node, the destination node is always assumed to be the identity node I without any loss of generality.

Definition 1 *[VS96] Consider an arbitrary node* $s = a_1 a_2 \cdots a_n$ *in* TCN_n*. There exists an unique integer* k *such that* $a_k = t_1^*$*. We define* **left distance** $D_L(s)$ *and* **right distance** $D_R(s)$ *of the node* s

(from the identity node) as follows:

$$D_L(s) = 2(k - m_1 - 1) + (n - k + 1)$$

$$D_R(s) = 2(n - k - m_2) + (k - 1)$$

where

$$m_1 = \max_m\{\exists(i,j) \mid (1 \le j \le k) \wedge (1 \le i \le n - m + 1) \wedge a_j a_{j+1} \cdots a_{j+m-1} = t_i t_{i+1} \cdots t_{i+m-1}\}$$

$$m_2 = \max_m\{\exists(i,j) \mid (k \le j \le n) \wedge (1 \le i \le n - m + 1) \wedge a_j a_{j+1} \cdots a_{j+m-1} = t_i t_{i+1} \cdots t_{i+m-1}\}$$

Then, we define the distance of the node s (from the identity node) as

$$D(s) = \min\{D_L(s), D_R(s)\}$$

Example 1: Consider the node $s = f\bar{g}hija\bar{b}cd\bar{e}$ in TCN_{10} (the identity node is $abcdefghij$). Here, $k = 6$, since $a_6 =$ "a" $= t_1$, $m_1(s) = 3$ (due to the substring "hij"), $m_2(s) = 2$ (due to the substring "cd"), $D_L(s) = 9$ and $D_R(s) = 11$. Hence $D(s) = 9$.

Theorem 1 *[VS96] For an arbitrary node $s = a_1 a_2 \cdots a_n$ in TCN_n, $D(s') = D(s) \pm 1$, where $s' = \delta(s)$ and $\delta \in \{g, f, g^{-1}, f^{-1}\}$.*

Definition 2 *Any cycle in TCN_n consisting of only the f-edges (induced by the symmetric functions f or f^{-1}) is called an f-cycle. Similarly, any cycle in TCN_n consisting of only the g-edges (induced by the symmetric functions g or g^{-1}) is called an g-cycle.*

Theorem 2 *All of the $n.2^n$ nodes of TCN_n of dimension n are partitioned into vertex disjoint g-cycles of length n; number of g-cycles in TCN_n is 2^n.*

Proof : Consider an arbitrary node $v = a_1 a_2 \cdots a_n$ in TCN_n. For any i, $i \ge 1$, let $g^i(v) = g(g^{i-1}(v))$, where $g^1(v) = g(v)$. It is easy to observe that $g^n(v) = v$. Also, $g^i(v) \ne g^j(v)$ for $1 \le i, j \le n$ and $i \ne j$. Thus, from an arbitrary vertex v if the g function is repeatedly applied, a cycle of length n is traced in the graph TCN_n. That these g-cycles are vertex disjoint follows from the fact that $g(v_1) = g(v_2)$, if and only if $v_1 = v_2$. □

Remark 2

- *Consider the symbol set* $\{t_1, t_2, \cdots, t_n\}$ *for* TCN_n*. For all* k*,* $1 \le k \le n$*, each* g*-cycle in* TCN_n *has a unique node starting with* t_k^* *(either* t_1 *or* $\bar{t}_1$*, but not both).*

- *For each* g*-cycle in* TCN_n*, the unique node starting with* t_1^* *is called the* leader *node. Since there are* n *symbols and the leader nodes start with* t_1^* *(either* t_1 *or* $\bar{t}_1$*), there are* 2^n leader *nodes in* TCN_n *which is equal to the number of* g*-cycles in* TCN_n*.*

- *Consider an arbitrary leader node* $t_1^* t_2^* t_3^* \cdots t_n^*$ *(of some* g*-cycle); each leader node maps to a* n *bit binary number by assigning* 0 *if* $t_i^* = \bar{t}_i$ *and* 1 *if* $t_i^* = t_i$ *for* $1 \le i \le n$*. This gives us a convenient way to number all the* 2^n g*-cycles in* TCN_n *from* g_0 *to* g_{2^n-1}*.*

Theorem 3 *For any arbitrary vertex* v *in* TCN_n *such that* $f(v) = u$ *and* $g(v) = w$*, there exists a vertex* x *such that* $g(x) = u$ *and* $f(x) = w$*; furthermore, the nodes* v*,* u*,* w *and* x *are all distinct and nodes* u *and* x *belong to the same* g*-cycle.*

Proof : Consider an arbitrary vertex $v = a_1 a_2 \cdots a_n$. Then $u = f(v) = a_2 \cdots a_n \bar{a}_1$ and $w = g(v) = a_2 \cdots a_n a_1$. Choose the node x as $x == g^{-1}(u) = \bar{a}_1 a_2 \cdots a_n$. Thus, $g(x) = u$ and $f(x) = a_2 \cdots a_n a_1 = w$. That these four nodes are distinct are also obvious from the fact that the different symbols in the nodes are distinct. □

Corollary 1 *Similarly, for any arbitrary vertex* v *in* TCN_n *such that* $f^{-1}(v) = u'$ *and* $g^{-1}(v) = w'$*, there exists a vertex* x' *such that* $g(u') = x'$ *and* $f(w') = x'$*; nodes* v*,* u'*,* w' *and* x' *are all distinct and nodes* u' *and* y' *belong to the same* g*-cycle, but different from the* g*-cycle containing nodes* u *and* y *of Theorem 3.*

Definition 3 *Two* g*-cycles, say* g_i *and* g_j*, are said to be adjacent if there exists a vertex* $v \in g_i$ *and a vertex* $u \in g_j$ *such that* $v = f(u)$ *or* $u = f(v)$*.*

Theorem 4 *If two* g*-cycles* g_i *and* g_j *are adjacent, then there are two f-edges connecting* g_i *and* g_j*.*

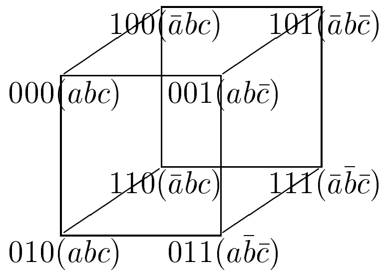

Figure 2: The reduced graph of TCN_3

Proof : Since g_i and g_j are adjacent, there exists a vertex $v \in g_i$ such that $f(v) = u$, where $u \in g_j$. Let $g(v) = w$. Then by Theorem 3 $f^{-1}(w) = y = g^{-1}(u)$; thus y is a node in g_j. Thus there are two f-edges between the two g-cycles g_i and g_j. □

Theorem 5 *Each g-cycle in TCN_n is adjacent to n different g-cycles.*

Proof : Consider an arbitrary g-cycle with the leader $v = a_1a_2 \cdots a_n$, where $a_1 = t_1^*$. Now, $f(v) = a_2a_3 \cdots a_n\bar{a}_1 = y_0$ and the node y_0 belongs to the g-cycle with leader $\bar{a}_1a_2a_3 \cdots a_n$. In general, consider the nodes y_i, $1 \leq i < n$, such that $y_i = f(v_i)$ where $v_i = g^i(v)$. We have $y_i = f(a_{i+1}a_{i+2} \cdots a_na_1a_2 \cdots a_i) = a_{i+2}a_{i+3} \cdots a_na_1a_2 \cdots a_i\bar{a}_{i+1}$; this node y_i belongs to a g-cycle with the leader $a_1a_2 \cdots a_i\bar{a}_{i+1}a_{i+2} \cdots a_n$. Obviously, the nodes y_i, $0 \leq i < n$, belong to different f-cycles (they have different leaders) and hence any g-cycle is adjacent to n different g-cycles in TCN_n. □

Corollary 2 *Consider a g-cycle g_i for a given i, $0 \leq i < 2^n$; i is a (n) bit binary number, say $b_{n-1}b_{n-1} \cdots b_0$. Then the g-cycle g_i or $g_{b_{n-1}b_{n-1} \cdots b_0}$ is adjacent to the following n g-cycles: $g_{\bar{b}_{n-1}b_{n-1} \cdots b_0}$, $g_{b_{n-1}\bar{b}_{n-1}b_{n-2} \cdots b_0}$, $\cdots$, $g_{b_{n-1}b_{n-1} \cdots \bar{b}_0}$.*

Definition 4 *For a given TCN_n compute the reduced graph RG_n with respect to g-cycles in the following way: condense each g-cycle into a single node and label that node with the n bit binary number corresponding to the g-cycle (see corollary 2); connect two arbitrary vertices by an undirected edge iff the corresponding g-cycles are adjacent in TCN_n.*

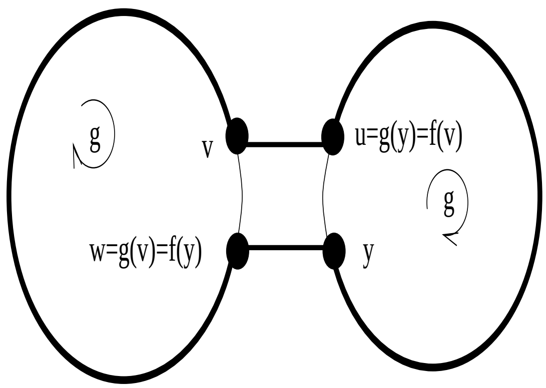

Figure 3: Combining two g-cycles to produce a larger cycle

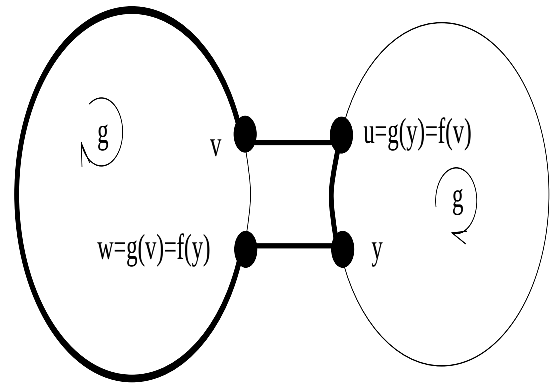

Figure 4: Combining a g-cycle with 2 adjacent nodes to produce a cycle of length $n+2$

Remark 3 *Figure 2 shows the reduced graph RG_3 corresponding to TCN_3. Each vertex in the reduced graph RG_n corresponding to TCN_n has a binary label of length n and has a degree n; its neighbors being all g-cycles whose labels are at a Hamming distance 1 from it by Corollary 2. Clearly, the reduced graph RG_n corresponding to TCN_n is a hypercube of order n [SS88].*

Theorem 6 *The reduced graph RG_n corresponding to TCN_n has a vertex connectivity n.*

Proof : RG_n is a hypercube of dimension n; a hypercube of dimension n has a vertex connectivity n; see [SS88]. □

Consider two arbitrary adjacent g-cycles, say g_1 and g_2. By Theorem 3, there exist nodes $v, w \in g_1$ and nodes $u, y \in g_2$ such that $u = g(y) = f(v)$ and $w = g(v) = f(y)$. A larger cycle can be constructed involving all nodes of g_1 and g_2 by using these two f-edges as shown in Figure 3. This, coupled with the facts that g-cycles in TCN_n are vertex-disjoint and each g-cycle is adjacent to exactly n distinct other g-cycles, immediately suggests a procedure to construct cycles in TCN_n of length $k * n$, $1 \leq k \leq 2^n$.

Remark 4 *Given any g-cycle (of length n), one can choose any two consecutive nodes, and apply Theorem 3 to generate a cycle of length $n+2$; see Figure 3. Thus, for two adjacent g-cycles, either we can combine the two g-cycles to generate a cycle of length $2n$ or we can combine one of the g-cycles with two nodes to generate a cycle of length $n+2$.*

Remark 5 *Combining the above two remarks, we have can identify cycles in TCN_n of length $kn + 2k'$ where k and k' are positive integers and $k + k' \leq 2^n$.*

Theorem 7 *In presence of a single arbitrary faulty node the graph TCN_n contains a cycle of length $N-1$, if n is odd and of length $N-2$ if n is even, where N is the number of nodes in TCN_n, $N = n2^n$.*

Proof : Let g^* be the g-cycle containing the faulty node; label the nodes in g^* as $u_1, u_2, \cdots, u_n$ and let u_1 be the faulty node without any loss of generality. The reduced graph RG_n without the g-cycle g^* is still connected and following the construction scheme discussed earlier, we have a cycle C of length $N-n$ consisting of all other g-cycles. The fault-free nodes in g^* can now be paired as (u_2, u_3), (u_4, u_5), $\cdots$. Now, $u_3 = g(u_2)$ and hence by Theorem 3, the nodes $f(u_2)$ and $f^{-1}(u_3)$ belong to the same g-cycle which is already in the large cycle C and hence the nodes u_2 and u_3 can be combined with C to produce a larger cycle of length $N - n + 2$ (Remark 4). We do the same for all the pairs. When n is odd, all fault free nodes of g^* can be combined while if n is even the last node u_n cannot be combined. Thus, the length of the resulting cycle is $N - 1$, if n is odd and is $N - 2$ if n is even. □

Figure 5 shows the cycle of length 23 in TCN_3 when the faulty node is bac.

Theorem 8 *In presence of two arbitrary faulty nodes the graph TCN_n contains a cycle of of length at least $N - 4$, where N is the number of nodes in TCN_n, $N = n2^n$.*

Proof : The proof is very similar to that of the previous theorem. If both faulty nodes belong to the same g-cycle g^*, we proceed as before to construct the large cycle containing all the rest of the g-cycles and then connect the non faulty nodes from g^* to the large cycle; if the faulty nodes are adjacent, the length of the resulting cycle would be either $N-2$ or $N-3$, depending on whether n is even or odd. If the faulty nodes are not adjacent, worst case length of the resulting cycle is $N - 4$. If the faulty nodes belong to two different g-cycles g_1^* and g_2^* respectively, we can treat each as a separate case like in the proof of the previous theorem. Note that the reduced graph RG_n is connected without the two g-cycles (connectivity of RG_n is n) and each node in any g-cycle is connected to two different nodes in two different g-cycles (by f and f^{-1} functions respectively). Thus, the argument is valid even when g_1^* and g_2^* are adjacent and the worst case length of the resulting cycle is $N - 4$. □

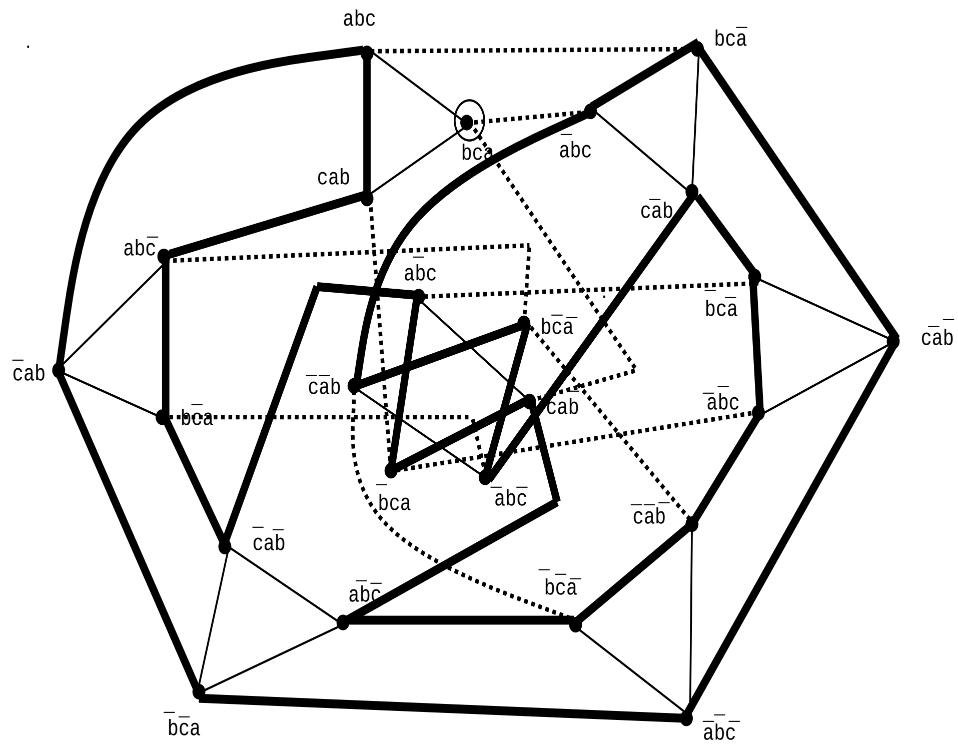

Figure 5: Cycle in presence of single fault

4 Conclusion

We have investigated different topological properties of the newly proposed Cayley networks of constant degree 4. Specifically, we have identified the cycle structure in TCN_n, and showed that TCN_n contains a Hamiltonian. We have shown that TCN_n contains a cycle of length $N-2$ ($N = n2^n$) in the presence of a single arbitrary faulty node and contains a cycle of length at least $N-4$ in presence of two arbitrary faulty nodes. Thus, the Cayley networks TCN_n are robust in terms of embedding fault tolerant rings.

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