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Abstract

We propose a new family of directed interconnection network graphs for an arbitrary number of nodes. The proposed network graph is almost regular (the difference between the in-degrees and out degrees of nodes is 2, a constant independent of the size of the network), has a diameter sub logarithmic in the number of nodes, is optimally fault tolerant and can be defined for an arbitrary number of nodes.

1 Introduction

Design of a communication network is an integral part of developing any distributed and parallel processing system. A communication network is usually modeled by a graph where the nodes (vertices) denote the computing elements and the the edges (arcs) denote the communication channels; if the channels are bidirectional, the graph is *undirected* and if the channels are unidirectional thy graph is *directed*. Desirable features for a good interconnection topology include properties like low degree, regularity, small diameter, high fault tolerance (connectivity), efficient routing algorithm, etc. The small diameter helps to keep the interprocessor communication delay low while the low degree of nodes is necessary to limit the number of input-output ports to some acceptable value. Many of the works in network topology has dealt with symmetric or undirected graphs [DT94, AK89, Pra85a, Pra85b], while others concentrated on directed graphs [RPK80, II83, ISO85, FLV88, FM88, FL92, Cor92, FMC93].

Most popular interconnection network has been the well known binary n-cubes or hypercubes; they have been used to design various commercial multiprocessor machines and they have been extensively studied. Recently there has been a spurt of research on Cayley graphs, symmetric graphs defined on permutation of distinct symbols. Most important of them are the so-called star graphs [AK89] which seem to enjoy most of the desirable properties of the hypercubes at considerably less cost; they accommodate more nodes with less interconnection hardware and less communication delay. Almost all research on Cayley graphs has been centered around undirected graphs [LJD93].

Only very recently Corbett [Cor92] has pointed to a family of directed Cayley graphs, network graphs based on permutation of elements, called the *rotator graphs*; it is to be noted these rotator graphs are in fact special case ($\Delta = D$) of the digraphs , $_\Delta(D)$, with order $(\Delta+1)\Delta\cdots(\Delta-D+2)$, proposed by Faber and Moore in [FM88] (see also [FMC93]). These rotator graphs are Cayley graphs except that the generators are not closed under inverse and hence the graph is directed (communication channels are unidirectional). These rotator graphs compete very favorably with the hypercubes and the star graphs in the sense that they have a lower diameter for the same number of edges and same number of nodes; this is a very desirable feature since the number of edges in the graph is directly related to the cost of the network. These rotator graphs are also significant in the sense that they are the only known directed graphs based on permutation of groups of symbols. But they can be defined for N nodes only when $N = n!$

for some integer n; they cannot be defined for an arbitrary number of nodes. This *incremental extensibility* is a very essential and desirable property in real life applications of a topology in designing computer networks. A few of the symmetric (undirected) network graphs described in the literature [AL82, FS81, BA84, SS91, SS92, LB94] are incrementally extensible. There are also incrementally extensible directed network graphs in the literature [II83, RPK80, FLV88, FL92].

Our purpose in the present paper is to propose another new incrementally extensible directed network topology that can be defined for an arbitrary number of nodes. The design philosophy basically involves appropriate interconnection of different sized rotator graphs of different sizes. If N is the given number of nodes, $n! < N < (n+1)!$, the proposed graph is a superset of several rotator graphs of size less than or equal to $n!$; we call it a super rotator graph. We prove that the new topology has the following characteristics: (1) the difference between the maximum and minimum in-degrees and out-degrees of all nodes is 2, a constant independent of the size of the network, (2) the diameter is *sub logarithmic*, (3) the network is maximally fault tolerant in the sense that the network remains strongly connected after $\delta - 1$ node failures where δ is the minimum in-degree or out-degree of a node in the graph, (4) the number of directed edges in the graph is $O(N\mathcal{F}(N))$ where $\mathcal{F}(N) = n$, iff $n! \leq N \leq (n+1)!$, and (5) addition of a new node in the existing graph is easy and simple. Note that some of the existing directed graphs, e.g. [FLV88, FL92] are also almost regular with good connectivities. It is also to be noted that we do not attempt to minimize the diameter of a directed graph with a given number of nodes; rather we propose a new family of graphs designed around the Cayley graphs; the proposed graphs compare favorably with the existing ones and addition/deletion of nodes to an existing network needs minimal or no reorganization.

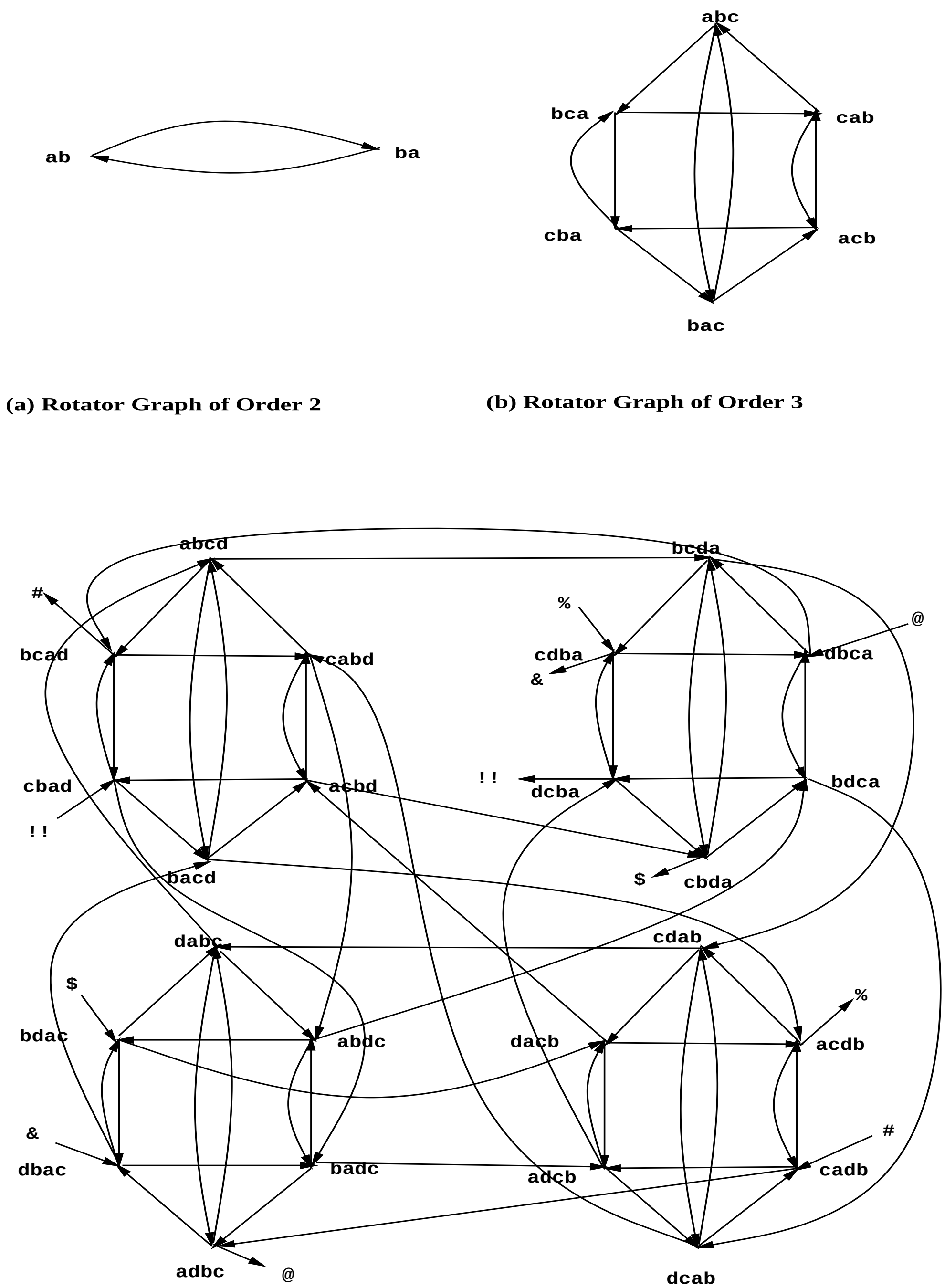
2 Basic Concepts

In this section we briefly introduce the *rotator graphs*, discuss relevant properties and introduce a few new concepts that will be needed to describe the new topology and to study its properties. Graph theoretic terms not defined here can be found in [Har72] and a detailed treatment of the rotator graphs can be found in [Cor92].

A rotator graph R_n, of order (dimension) n, is defined to be a directed graph $G = (V, E)$, where V is the set of $n!$ vertices (nodes), each representing a distinct permutation of n distinct symbols, and E is the set of directed edges such that there is an edge from one permutation (node) v to another permutation (node) u iff u can be reached from v by rotating its first ℓ symbols one place left ($2 \leq \ell \leq n$). These rotator graphs are Cayley Graphs [ABR90] except that the generators (different values of ℓ give the distinct generators) are not closed under inverse operation and hence the graph is a directed graph. For example, in R_3, the node *abc* has outgoing edges to nodes *bca* and *bac* as well as has incoming edges from nodes *bac* and *cab*. Figure 1 shows the rotator graphs of order 2, 3 and 4. We denote the nodes as permutations of English alphabets; for example, the identity permutation is denoted by $I = (abc...z)$ (z is the last symbol in the string, not necessarily the 26th letter). It has been shown in [Cor92] that: (1) R_n is $(n-1)$ regular in the sense that each node has both an in-degree and an out-degree of $n-1$ (hence, the number of edges is $n!(n-1)$), (2) the diameter of R_n is given by $\mathcal{D}(R_n) = n - 1$, and (3) R_n is vertex symmetric for all values of n like other Cayley graphs [AK89].

It is also to be noted that the rotator graphs are hierarchical in the sense that R_n can be decomposed into n number of R_{n-1}'s. In R_n we use V_x to denote the set of nodes (permutations) that end with the symbol "x"; obviously V_x is a rotator graph of dimension $n-1$. Similarly, we use V_α to denote the set of nodes that end with α where α represents a sequence of symbols. V_α is a rotator graph of dimension $n - |\alpha|$ if V_α is a subgraph of R_n.

Definition 1 *Consider any two mutually disjoint subgraphs V_x and V_y of a rotator graph R_n. The nodes of V_x that are directly connected to some node of V_y by outgoing edges are called the type I gateway nodes of V_x with respect to V_y; we denote this set of nodes by $G^I_{x,y}$. The nodes of V_x that are directly connected to some node of V_y*



(a) Rotator Graph of Order 2

(b) Rotator Graph of Order 3

(c) Rotator Graph of Order 4

Figure 1: Rotator Graphs of Dimensions 2, 3, and 4

by incoming edges are called the type II gateway nodes of V_x *with respect to* V_y*; we denote this set of nodes by* $G^{II}_{x,y}$.

Example: Consider the rotator graph R_4 in Figure 1. $G^{I}_{a,b} = \{bcda, bdca\}$, $G^{II}_{a,b} = \{cdba, dcba\}$, $G^{I}_{b,a} = \{adcb, acdb\}$, and $G^{II}_{b,a} = \{cdab, dcab\}$.

Definition 2 *A directed graph G is called* **strongly connected** *iff for an arbitrary pair of vertices u and v, there exists a directed path from u to v in G.*

Definition 3 *A directed graph G is called strongly k-connected if it remains strongly connected after removal of an arbitrary set of k or less nodes. This k is called the* **measure of strong connectedness** ξ *of the graph G.*

Remarks:

- A strongly 0-connected graph is simply a strongly connected directed graph.
- The rotator graph R_n is strongly $(n-2)$-connected, i.e., $\xi(R_n) = n-2$, for all $n \geq 3$.
- The measure of strong connectedness of a directed graph defines the node fault tolerance of the graph. A directed graph G, $\xi(G) = k$ remains strongly connected when an arbitrary set of k or less nodes are faulty.

Definition 4 *Any positive integer* N, $n! \leq N < (n+1)!$, *can be expressed in its mixed-radix form as* $< a_n, a_{n-1}, \cdots, a_1 >$, *where*

$$N = a_n.n! + a_{n-1}.(n-1)! + ... + a_1.1!$$

and $0 \leq a_i \leq i$ *for* $i = 1, \cdots, n-1$, *and* $0 < a_n \leq n$.

For example, $110 =< 4, 2, 1, 0 >$, since $4.4! + 2.3! + 1.2! + 0.1! = 96 + 12 + 2 = 110$.

In order to design the proposed super rotator graphs we need two types of connections between rotator graphs of different dimensions. We define them as follows.

Definition 5 *Given m copies of* R_k *where* $m \leq k$*, we say that these m copies are joined by type A connections when they are connected by the directed edges as if they were subgraphs of the larger* R_{k+1}.

Remark: There are exactly $(k-1)!$ directed type-A edges from each R_k to each of the other R_k's; similarly there are exactly $(k-1)!$ directed type-A edges from each of the other R_k's to a specific R_k. Figure 2 shows the type-A edges between two copies of R_3.

Definition 6 *When m copies of* R_k, $m \leq k$, *are joined by the type A connections, the resulting graph is called a class* $C_k(m)$.

Example: Figure 2 shows a $C_3(3)$. For a given class $C_k(m)$, we arbitrarily number the m components from 1 to m as $C^{\ell}_k(m)$, $1 \leq \ell \leq m$, and we call the first component $C^1_k(m)$ the **leader** $L_k(m)$ of the class $C_k(m)$.

Definition 7 *Given two rotator graphs* R_m *and* R_n, $m < n$, *they are said to be joined by type B connections if outgoing edges are added from each node w in* R_m *to* $|n-m|$ *different nodes of* R_n *(type B successors of w in* R_n*) and incoming edges are added to each node w in* R_m *from* $|n-m|$ *different nodes of* R_n *(type B predecessors of w in* R_n*). It is required that type B successors of an arbitrary pair of nodes in* R_m *are mutually disjoint and so are their type B predecessors; but type B successors and type B predecessors of nodes may overlap.*

Example: Figure 3 shows the type B connections between a R_3 and a R_1.

Vertex Numbering: It is well known [Knu72] that all the $n!$ permutations of n distinct symbols can be uniquely numbered from 0 through $n!-1$. We use this scheme to number the vertices of any rotator graph R_n; we also extend this scheme to number the vertices of a class $C_k(m)$. The class $C_k(m)$ has $m.k!$ nodes; the nodes of C^1_k are numbered from 0 to $k!-1$, the nodes of C^2_k are numbered from $k!$ to $2k!-1$ and so on.

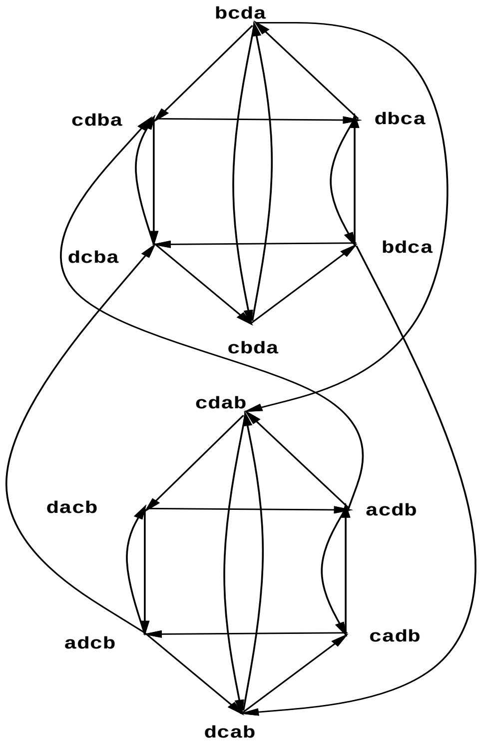


Figure 2: Type A edges between two R_3

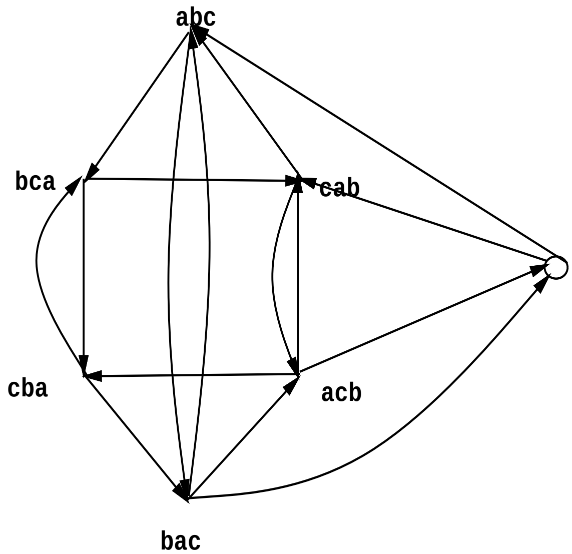


Figure 3: Type B edges between R_3 and R_1

3 Topology for Super Rotator Graphs

The basic idea behind the design of the super rotator graph of N nodes, when $n! < N < (n+1)!$, is to express N as sum of several factorials, build smaller rotator graphs of appropriate dimensions, and then add appropriate type A and type B edges to connect those smaller graphs. The following algorithm builds the super rotator graph for any given N, $n! < N < (n=1)!$.

The Algorithm

Step 1: *[Build the smaller rotator subgraphs]*

Compute the mixed radix representation of $N =< c_n, c_{n-1}, \cdots, c_1 >$ and construct c_i copies of R_i for all i, $1 \leq i \leq n$ (note $c_n \neq 0$).

Step 2: *[Label the nodes]*

- Choose $n+1$ symbols to label the nodes (permutations). We use $n+1$ consecutive English letters starting with "a".
- For $i = n$ to 1 do the following (fix the i-th symbol for the nodes):
  - if $c_i \neq 0$ then label each of the c_i copies of R_i as $V_{\alpha_j \beta}$ where $\beta = symbol(i+1)symbol(i+2)\cdots symbol(n)$, and α_j, $1 \leq j \leq c_i$, are chosen in alphabetic order from the set of symbols that are yet to be allocated to the "symbol" array.
  - Set $symbol(i)$ to be equal to the next available English letter in alphabetic order.

Step 3: *[Provide type A connections among rotator subgraphs to form classes*

- For each i, $1 \leq i \leq n$, join the c_i components of R_i's by type A connection as defined earlier to get the different classes C_i (note that this does not connect the rotator subgraphs of different dimensions).
- Each class C_i has c_i number of components C_i^ℓ, $1 \leq \ell \leq c_i$ each of which is a rotator graph of dimension i. The vertices in C_i are numbered from 0 to $c_i.i! - 1$ by using the vertex numbering scheme as described before (the vertices of C_i^1 are numbered from 0 to $i! - 1$, those of C_i^2 are numbered from $i!$ to $2i! - 1$ and so on).

Step 4: *[Construct the super rotator graph in steps by providing the type B connections]*

Find the minimum i such that $c_i \neq 0$ and then set $j = i$ and set $SR_j = C_i$ (SR_j denotes the super rotator graph with $\sum_{k=1}^{j} c_k k!$ nodes).
while $i \leq n$ do
if $c_i \neq 0$ then

- Establish type B connections between SR_j and C_i. Each node in SR_j is assigned $(i-j)$ type B successors as well as $(i-j)$ type B predecessors in the leader L_i of the class C_i. This is easily done by using the node numberings in both the graphs SR_j and C_i (e.g., outgoing edges are introduced from node "0" of SR_j to nodes "0" through "i-j-1" of L_i as well as incoming edges are introduced from nodes "$i! - 1$" to "$i! - (i-j)$" of L_i to node "0" of SR_j, outgoing edges are introduced from node "1" of SR_j to nodes "i-j" through "2(i-j)-1" of L_i as well as incoming edges are introduced from nodes "$i! - (i-j) - 1$"*to*"$i! - 2(i-j)$" of L_i to node "0" of SR_j, and so on).
- Renumber the nodes of SR_j by adding $c_i i!$ to each node number.
- Set $j = i$ and set SR_j to be the composite graph generated in the previous steps. Note that SR_j has now $\sum_{k=1}^{j} c_k k!$ nodes and they are numbered from 0 to $\sum_{k+1}^{j} c_k k! - 1$.

$i = i + 1$

Return SR_n as the desired super rotator graph of N vertices.

Remarks:

- In step 4, for each i, whenever type B connections are provided between a leader L_i of a class C_i and some smaller super rotator SR_j, $j < i$, the in-degree of each node of L_i is increased *at most* by 1; and the out-degree of each node of L_i is also increased *at most* by 1. This is evident from three facts: (1) a leader L_i is a rotator graph R_i of $i!$ nodes, (2) the maximum number of nodes in the super rotator graph SR_j is $(j+1)! - 1$, and (3) $i! > (i-j)\{(j+1)! - 1\}$ for any integer i and j, $i > j$.
- In step 4, for each i, SR_i represents a super rotator graph of $\sum_{k=1}^{i} c_k k!$ vertices.

Example 1: Let $N = 13$. Then N can be expressed as $N =< 2, 0, 1 >$ or $c_3 = 2, \quad c_2 = 0, \quad c_1 = 1$. Here $n = 3$ and there are three classes, e.g., C_3, C_2 and C_1 of which C_2 is null since $c_2 = 0$. C_3 has two components: $C_3^1 = V_a$ which is also the leader of this group, and $C_3^2 = V_b$; each of these components is a rotator graph of dimension 3. See Figure 4. The nodes of C_3 are numbered from 0 to $2.3! - 1 = 11$; the numberings are shown in parenthesis in the figure. Thus $symbol(3) = c$ and in the next step we get $symbol(2) = a$, since $c_2 = 0$. Hence $symbol(1) = b$ and the class C_1 is a rotator graph of dimension 1, i.e., a single vertex "*dbac*". Type A connections are provided in C_3 by joining all the gateway points to their counterparts in the components. To provide the type B connections, we add two directed edges from the node 0 of class C_1 to the nodes 0 and 1 of C_3^1, the leader of the class C_3; and add

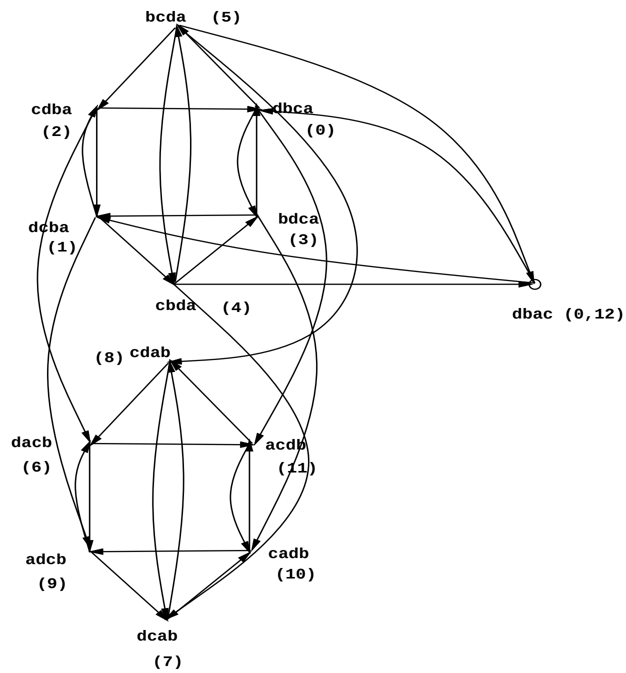


Figure 4: Super Rotator Graph with $N = 13$ nodes

two directed edges from nodes 5 and 4 of C_3^1 to the node 0 of class C_1. Lastly, the node "*dbac*" of C_1 is renumbered as 12 by adding 2.3! to its original numbering.

Example 2: Let $N = 23$. Then N can be expressed as $N =< 3, 2, 1 >$ or $c_3 = 3$, $c_2 = 2$, and $c_1 = 1$. We have 3 non-null classes. See Figure 5. The class C_3 have 3 components, e.g., V_a, V_b and V_c each of which is a rotator graph of dimension 3; the vertices are numbered from 0 to 17. Also, $symbol(3) = d$. The class C_2 has two components V_{ad} and V_{bd} each of which is a rotator graph of dimension 2; the vertices are numbered from 0 to 3. As before, the class C_1 is a single node (a rotator graph of dimension 1) and since $symbol(2) = c$, this single node is labeled as the permutation "*bacd*" and is numbered 0. Type A connections are provided in each class as shown in the figure. In the first iteration of step 4 of the design algorithm, type B connections are provided to nodes of C_1 and C_2 by adding directed edges from node "0" of C_1 to node "0" of C_2 and from node "1" of C_2 to node "0" of C_1 and we get SR_2. Nodes of SR_2 are renumbered (actually the nodes of C_1 only need be renumbered; the node "*bacd*" is renumbered as 4). Next, C_3 and SR_2 are joined by type B connections to get the desired super rotator graph $SR_3(23)$ (nodes "0" through "4" of SR_2 are connected to "0" through "4" of C_3 as well as nodes "5" through "1" of C_3 are connected to "0" through "4" of SR_2).

4 Properties of the Super Rotator Graphs

In this section we develop interesting algebraic properties of the super rotator graphs $SR_n(N)$, where N is the number of nodes in the graph and $n! < N < (n+1)!$. We use ξ and $\mathcal{D}$ to indicate the measure of strong connectedness (node fault tolerance) and the diameter respectively of a directed graph.

Lemma 1 *When m copies of R_n, $m \leq n$, are connected by type A connections to form a class $C_n(m)$, $\xi(C_n(m))$ is given by $n - 2$.*

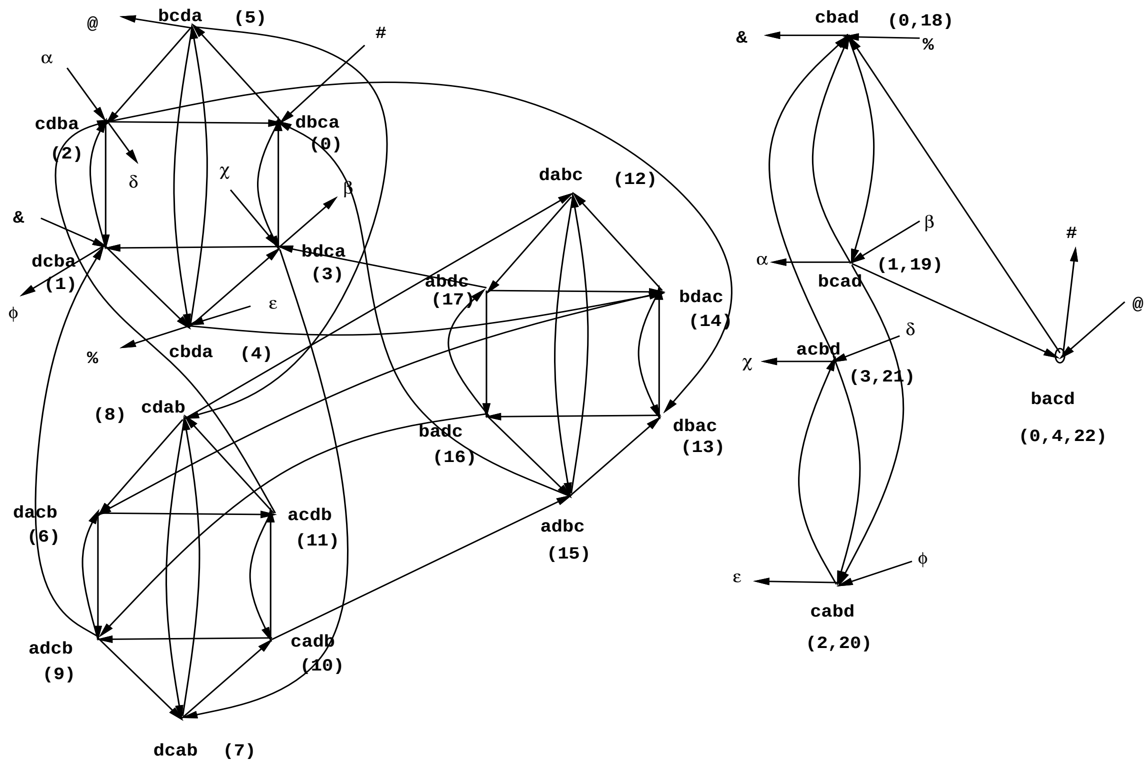


Figure 5: Super Rotator Graph with $N = 23$ nodes

Proof : $C_n(m)$ is made of m components C_n^ℓ, $1 \le \ell \le m$ each of which is a rotator graph of dimension n, i.e., each component of the class is strongly $(n-2)$-connected. Assume an arbitrary set of $n-2$ nodes to be faulty. Each component of the class is still strongly connected. Consider any two components; there are $(n-1)!$ incoming edges as well as $(n-1)!$ outgoing edges; no two of these edges are parallel and also $(n-1)! > n-2$. Thus all the components are strongly connected among each other by the type A edges. Hence the proof. □

Lemma 2 *When two rotator graphs R_m and R_n, $m < n$, are connected by type B connections, then the resulting directed graph is strongly $(n-2)$-connected.*

Proof : For an arbitrary set of $(n-2)$ faulty nodes, we have to show that the combined graph is still strongly connected. If all the faulty nodes are in R_n, the proof is trivial since the rotator graph R_n is strongly $(n-2)$-connected. If there are less than $(m-2)$ faulty nodes in R_m and the rest are in R_n, both R_m and R_n are strongly connected and hence the combined graph as well. If there are $(m-2)$ or more faulty nodes in R_m, then R_m may be disconnected. Consider any surviving node v of R_m; v has $(n-m)$ type B successors as well as $(n-m)$ type B predecessors in R_n; at least one successor and at least one predecessor must be fault free. Thus R_n is strongly connected and each surviving node of R_m is strongly connected to R_n. Hence the proof of the lemma. □

Theorem 1 *The super rotator graph $SR_n(N)$ of N nodes, $n! \le N < (n+1)!$, is strongly $(n-2)$-connected, i.e., $\xi(SR_n(N)) = n-2$.*

Proof : The proof follows from step 4 of the design algorithm. At the beginning of step 4, we have different C_i's (for different values of i) of strong connectedness measure $i-2$. The smallest initial super rotator graph is a class.

Whenever we join C_i with a super rotator graph SR_j, $j < i$ by type B connections, the resulting graph SR_i has a strong connectedness measure of $i - 2$ by the previous lemmas. Hence, when the algorithm terminates, the final super rotator graph $SR_n(N)$ has a strong connectedness measure of $n - 2$. $\square$

Corollary 1 *The minimum in-degree (out-degree) of any node in $SR_n(N)$ is $n - 1$.*

Remark: A super rotator graph of N vertices, where $n! \leq N < (n+1)!$ has the same strong connectedness measure as a rotator graph R_n of $n!$ vertices.

Theorem 2 *The diameter of a class $C_i(m)$ (m copies of rotator graphs R_i joined by type A connections) is upper bounded by $\mathcal{D}(C_i(m)) \leq 2i - 1$.*

Proof : Consider two arbitrary nodes u and v in $C_i(m)$. If both u and v belong to the same component C_i^ℓ $(\ell \leq m)$, there is a directed path from u to v of length $\leq i - 1$ (since $\mathcal{D}(R_i) = i - 1$; each component is a rotator graph of dimension i). If u and v belong to different components, say V_x and V_y, and there are no other components in the class, then we must reach from u a gateway point in $G^I_{x,y}$ (any node in $G^I_{x,y}$ has "y" as the first symbol and "x" as the last symbol) and then go to a node in $G^{II}_{y,x}$ and then go to v in V_y; this is true since we cannot apply the generator g_i until we are at a gateway point since there is no other component. Thus the worst case distance between u and v is given by $(i - 1) + 1 + (i - 1) = 2i - 1$. Hence the theorem. $\square$

Remark: Consider the example 1 in Figure 4. C_3 has two components V_a and V_b. Let $u =$ "cdba" $\in V_a$ and $v =$ "acdb". The minimal path between u and v is given by $u = cdba \rightarrow dbca \rightarrow bcda \rightarrow cdab \rightarrow dacb \rightarrow acdb = v$ and the length is $5 = 2.3 - 1$.

Theorem 3 *The diameter of the super rotator graph $SR_n(N)$, $n! < N < (n + 1)!$, is upper bounded by $\mathcal{D}(SR_n(N)) \leq 2n$.*

Proof : Consider two arbitrary nodes u and v. We need to consider the following cases:
Case 1: $u \in C_i$, $v \in C_j$ and $i = j \leq n$. Then $d(u, v)$ is upper-bounded by $2i - 1$ (theorem 2).
Case 2: $u \in C_i$, $v \in C_j$ and $i \neq j$. Assume $j < i$ without loss of generality. The node $v \in C_j$ has at least one type B successor $v_{succ} \in C_i$ and at least one type B predecessor $v_{pred} \in C_i$. By theorem 2 we can go from v_{succ} to u by at most $2i - 1$ hops and from u to v_{pred} by at most $2i - 1$ hops. Hence, the distance between the nodes u and v is either way upper-bounded by $2i - 1 + 1 = 2i$. Hence the theorem. $\square$

Theorem 4 *Total number of edges in a super rotator graph $SR_n(N)$, $n! < N < (n + 1)!$ and $N =< c_n, \cdots, c_1 >$, is given by*

$$\sum_{i=1}^{n} \left\{ c_i(i - 1)i! + 2\binom{c_i}{2}(i - 1)! + 2c_i(n - i)i! \right\} = \mathcal{O}(Nn)$$

Proof : The first term in the expression counts the total number of directed edges in all the complete rotator graphs (components of the different classes). The second term accounts for all the type A edges; for any class $C_i(c_i)$ $(c_i > 0)$ there are $\binom{c_i}{2}$ pairs of component rotator graphs of dimension i and for each pair there are $(i - 1)!$ directed type A edges in either direction. The third term accounts for all the type B edges ; each class C_i, $i \neq n$, has $c_i i!$ nodes; each node is connected to $(n - i)$ nodes in the leaders of higher order classes by directed edges as well as has $(n - i)$ directed type B edges incident on it from the leaders of higher order classes. $\square$

Theorem 5 *The maximum in-degree (out-degree) of any node in a super rotator graph $SR_n(N)$ is $n + 1$.*

Proof : Any node in the class C_i has a maximum in-degree (out-degree) of $i+1$ (since any node of a rotator graph R_i has in-degree (out-degree) of i and in a class any node can have at most one type A incident edge and at most one type A outgoing edge. Each node in a class C_i has $(n-i)$ type B incoming edges from and $(n-i)$ type B outgoing edges to higher order classes and can have *at most* one incoming type B edge from and *at most* outgoing edge to lower order classes. Hence the maximum in-degree (out-degree) of any node in a super rotator graph is $n+1$. □

Definition 8 *Consider any arbitrary directed graph G where the minimum of the in-degree and out degree of any node is* δ*. The graph G is called* **maximally fault tolerant** *iff G is strongly* $(\delta-1)$*-connected; the fault tolerance is maximal since the graph's connectivity cannot exceed this value.*

Rotator graphs are shown to be maximally fault tolerant [Cor92]; each node in R_n has in-degree (out-degree) of $n-1$ and R_n is strongly $(n-2)$-connected. In the following we show that the super rotator graphs are also maximally fault tolerant in the same sense.

Lemma 3 *For any super rotator graph* $SR_n(N)$, $n! < N < (n+1)!$, *there exists at least one node with in-degree or out-degree* $n-1$.

Proof : Consider the leader L_n of the class C_n; this leader is a rotator graph of dimension n with $n!$ nodes while the rest of the graph $SR_n(N)$ excluding this class C_n has at most $n!-1$ nodes. Thus, at least one node in L_n has no incoming type B edge as well as at least one node with no outgoing type B edge (these two nodes may not be the same). So, there is at least one node for which minimum of the in-degree and the out-degree is $n-1$. □

Theorem 6 *The super rotator graph* $SR_n(N)$ *is maximally fault tolerant.*

Proof : The proof readily follows from the facts that $SR_n(N)$ is strongly $(n-2)$-connected and the minimum in-degree (out-degree) of a node is $n-1$. □

5 Conclusion

We have proposed a new class of directed network graphs that can be effectively used in designing the communication architecture for distributed processing systems. The design of the graphs is based on the theory of Cayley graphs. The proposed family of graphs has the following interesting properties:

- The graph can be easily defined for any given number of nodes.
- The graph has a sub logarithmic diameter and is maximally fault tolerant in the sense that the graph remains strongly connected when the number of faulty nodes is less than the minimum of the in-degree and out degree of any node.
- The difference between the maximum in-degree (out-degree) and the minimum in-degree (out-degree) of nodes is 2, a constant independent of the number of nodes, i.e., the graph is *almost regular*.
- The number of directed edges in the graph is $O(N\mathcal{F}(N))$ where $\mathcal{F}(N)=n$, iff $n! \leq N \leq (n+1)!$.
- Additional nodes can be added to an existing graph with no or minimal reorganization of the existing interconnections.

Investigations are underway to design optimal routing algorithms as well as the fault tolerance and diagnosability issues of these graphs.

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