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The $\mathcal{Z}$-Polyhedral Model*

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Abstract

The polyhedral model is a well developed formalism for the specification, analysis and transformation of regular loop programs. The model has been extensively used in a variety of contexts *viz.* automatic parallelization of loop programs, locality, program verification, hardware generation and more recently, automatic reduction of asymptotic program complexity. Such analyses and transformations are based on certain closure properties in the polyhedral model. However, the polyhedral model is limited in expressivity and the need for the extension to a more general class of programs is widely known.

We provide the extension to $\mathcal{Z}$-polyhedra which are the intersection of polyhedra and lattices. We prove the required closure properties using a novel representation and interpretation of $\mathcal{Z}$-polyhedra. In addition, we also prove closure in the $\mathcal{Z}$-polyhedral model under images by dependence functions—thereby proving that unions of LBLs, widely assumed to be a richer class, is equal to unions of $\mathcal{Z}$-polyhedra. These closure properties constitute the foundations of the $\mathcal{Z}$-polyhedral model. As an example, we present the automatic reduction of complexity in the $\mathcal{Z}$-polyhedral model.

1 Introduction

The polyhedral model provides sophisticated analysis and transformations of the kernels of many compute- and data- intensive applications. Programs in the polyhedral model essentially comprise of (i) variables (and expressions) representing collections of values defined over polyhedral domains, and (ii) affine dependences between computations. Feautrier [6] showed that an important class of conventional imperative loop programs called *affine control loops* (ACLs) can be transformed to programs in the polyhedral model. Significant parts of the SpecFP and PerfectClub benchmarks are ACLs [2].

An intuitive and general way specifying these programs is through a finite list of high level (mutually recursive) equations. For example, the following recurrence [3] for computing the cost of optimal string parenthesization is a program in the polyhedral model (of course, clothed in syntactic sugar).

$$C_{i,j} = \begin{cases} i = j & : \quad 0 \\ i < j & : \quad \min\limits_{i \le k < j} (C_{i,k} + C_{k+1,j} + f(i,j,k)) \end{cases}$$

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In the example, the dependence between computations is such that the variable C at the iteration at (i, j) depends on a range of values: between the iterations (i, k) and $(k+1, j)$, and on the value of f at the iteration (i, j, k). A complete description of the equational language is presented in section 2.

Many computations can be expressed naturally in the polyhedral model, *e.g.*, matrix multiplication, LU-decomposition, Cholesky factorization, Kalman filtering, as well as algorithms arising in RNA secondary structure prediction.

The polyhedral model has been used for the automatic parallelization of ACLs because it enables sophisticated analyses and transformations such as scheduling [4] and semantic-preserving refinements into (sequential or parallel) code [15, 1]. One can also automatically and optimally decrease the complexity of accumulations (called *reductions*) in the polyhedral model [8].

In an ACL (or the corresponding program in the polyhedral model), statement level transformations often yield better results than loop level transformations[**?**]. This relies on the closure of (statement/equation) domains under images by (a restricted class of) affine functions. Indeed, most analyses/transformations rely on closure properties in the polyhedral model. Also, as a result of these closure properties, we may design the equational language mentioned above such that the domain of every (sub)expression is of the same type as the domains of variables, permitting a unified framework based on expressions defined over polyhedral domains[1]. As a direct application, the reduction of asymptotic program complexity requires the closure of domains of expressions under set difference.

The polyhedral model currently suffers from certain limitations (although partial solutions, some ad-hoc, have been proposed for many of them).

- Loop programs with a non-unit stride, and *non-unimodular* transformations fall outside the scope of the model. This is an important class of programs [17, 13, 21, 7] arising in situations such as the red-black SOR computation for solving partial differential equations.
- Parallel architectures with *periodic* processor activity, such as *multi-rate arrays* [12] and bidirectional systolic arrays, cannot be described in the model.
- Reduction operations with arbitrary projections cannot be expressed in the model (explained later).

As a simple example consider the following equation for $i \geq 1$.

$$A[i] = \begin{cases} i \text{ even} & : \quad A[i/2] \\ i \text{ odd} & : \quad 0 \end{cases}$$

The two branches define the variable, each one on a subdomain that has *holes*, more precisely, on a $\mathcal{Z}$-polyhedron which is the intersection of an integer polyhedron and an *affine*

[1]In fact variables may be treated simply as named expressions.

integer lattice. Unfortunately, there does not exist an equivalent program in the polyhedral model (where variables have polyhedral domains, without *holes*) which captures such a dependence pattern.

It has been long claimed that these limitations can be resolved through some (deceptively, as it turns out) simple extensions to polyhedra. The first extension, proposed by Teich and Thiele, was called Linearly Bounded Lattices (LBLs) [19] which are images of integer polyhedra by arbitrary affine functions. The main motivation for LBLs was that polyhedral domains were not closed under images. Le Verge, in an unfinished manuscript [11] showed the limitations of LBLs and promoted $\mathcal{Z}$-polyhedra. He also showed that there is a strict inclusion: Integer Polyhedra $\subset \mathcal{Z}-$Polyhedra $\subset$ LBL. Till date, previously known theory and techniques of the polyhedral model—most notably program analysis and precise determination of regions of definition and usage of values—have not been extended to any of the richer models.

In this paper, we present the foundations of a simple and unified solution to all of these limitations through the extension of the equational language to domains that are unions of $\mathcal{Z}$-polyhedra. Our key insight is a new representation of $\mathcal{Z}$-polyhedra, and its associated interpretation. Specifically, we view any point in a $\mathcal{Z}$-polyhedra as the integer linear combinations of the generators of the associated lattice. The coefficients of these linear combinations are called *coordinates* which belong to a polyhedron. The critical hindrance in previous attempts was that $\mathcal{Z}$-polyhedra were either viewed as restricted images of integer polyhedra or as the intersection of integer polyhedra and affine lattices, rather than the coordinate view proposed here. Our key contributions are as follows.

- We present a novel representation for $\mathcal{Z}$-polyhedra and an associated family of functions, together with proofs of closure of unions of $\mathcal{Z}$-polyhedra under intersection, finite union, difference and preimage by the family of functions.

- We prove closure of unions of $\mathcal{Z}$-polyhedra under image by the family of functions, which had been a major limitation of the polyhedral model. This proves that unions of LBLs, widely assumed to be a richer class of domains, are equivalent to unions of $\mathcal{Z}$-polyhedra. This result relies on our theorem that weakens the sufficient conditions required to verify in polynomial time that an LBL is in fact a $\mathcal{Z}$-polyhedron .

- As an example, we present the automatic decrease of complexity of programs in the $\mathcal{Z}$-polyhedralmodel. This is done by transforming the analysis to an equivalent analysis on polyhedra. This shows that, often, tools and techniques developed for the polyhedral model can be reused.

The remainder of this paper is organized as follows. In the following section, we describe a generic equational language where expressions are associated with domains. The section shows the precise closure properties needed to ensure the semantic soundness of the equational language. The mathematical background on lattices, polyhedra, affine functions, etc., is described in section 3. In section 4, we present the polyhedral model as an instance of the generic equational language, and elaborate on the limitation on reductions. Our main results about the new representation and closure properties are described in Section 5. All our proofs

Expression	Syntax	Domain
Constants	Constant name or symbol	$\mathcal{D}_C$
Variables	V	$\mathcal{D}_V$
Operators	$\mathtt{op}(\mathtt{Expr}_1, \ldots, \mathtt{Expr}_M)$	$\bigcap_{i=1}^{M} \mathcal{D}_{\mathtt{Expr}_i}$
Case	$\mathtt{case}\ \mathtt{Expr}_1; \ldots ; \mathtt{Expr}_M\ \mathtt{esac}$	$\biguplus_{i=1}^{M} \mathcal{D}_{\mathtt{Expr}_i}$
Restriction	$\mathcal{D}' : \mathtt{Expr}$	$\mathcal{D}' \cap \mathcal{D}_{\mathtt{Expr}}$
Dependence	$\mathtt{Expr}.(z \rightarrow f(z))$	$f^{-1}(\mathcal{D}_{\mathtt{Expr}})$
Reductions	$\mathtt{reduce}(\oplus, (z \rightarrow f(z)), \mathtt{Expr})$	$f(\mathcal{D}_{\mathtt{Expr}})$

Table 1: Expressions: Syntax and Domains. If `op` is a binary operator, it may be written in infix notation. $\biguplus$ denotes disjoint union and f^{-1} denotes relational inverse.

are constructive, and should be accessible to a reader with a background in linear algebra, and for this reason we have chosen *not* to relegate the proofs to an appendix, although they may be skipped or skimmed on first reading. Then, we present the automatic and optimal decrease of the complexity of reductions in the $\mathcal{Z}$-polyhedral model. Finally, we discuss future and related work and present our conclusions.

2 Equational Language

Programs are a finite list of equations of the form `Var` = `Expr` where `Var` and `Expr` denote mappings from their domains to a set of values. The elements of a domain are called iteration points.

Expressions are constructed by the rules given in table 1 (column 2). The domains of all variables and constants are declared and the domains of expressions are derived (table 1 column 3). We adopt the convention that the domain of an expression A is denoted by $\mathcal{D}_A$, and the function $z \rightarrow f(z)$ by f. The function specified in a dependence expression is called the *dependence function* and the function specified in a reduction is called the *projection function*.

For compound expressions to be defined over the same family of domains, say $\mathcal{F}_{\mathcal{D}}$, all syntax rules must maintain closure with respect to $\mathcal{F}_{\mathcal{D}}$. Thus, $\mathcal{F}_{\mathcal{D}}$ must be closed under intersection, finite union, difference and image and preimage by the family of functions, say $\mathcal{F}_f$.

2.1 Semantics

Here, we provide the semantics of expressions over their domains of definition. At the iteration point z in its domain, the value of

- a constant expression is the associated constant.

- a variable is either provided as input or given by an equation; in the latter case, it is the value, at z, of the expression on its *rhs*.
- an operator expression is the result of applying `op` on the values, at z, of its expression arguments.
- a case expression is the value at z of that alternative, to whose domain z belongs. Alternatives of a case expression are defined over disjoint domains. This can be derived from a more general description in which the domains of the alternatives are non-disjoint, but are evaluated one after the other, since $\mathcal{F}_\mathcal{D}$ is closed under difference.
- a restriction over E is the value of E at z.
- the dependence expression $E.f$ is the value of E at $f(z)$.
- `reduce`$(\oplus, f, E)$ is the application of $\oplus$ on the values of E at all iteration points in $\mathcal{D}_E$ that map to z by f. $\oplus$ is an associative and commutative binary operator and therefore we may choose any order of its application.

It is often convenient to have a variable defined either entirely as input, or only by an equation. The former is called an *input variable* and the latter is a *computed variable*. Computed variables are just names for valid expressions. Since $\mathcal{F}_\mathcal{D}$ is closed under difference, it is always possible to transform any specification to have only input and computed variables.

2.2 Context Domain

Consider the set of iteration points at which the value of an expression is needed. This set is called the *context domain* of an expression [5, 8]. We can always transform an specification by restricting any (sub)expression to its context domain. Therefore, for closure, we require that the context domain also belongs to the family of domains.

The context domain of an expression is calculated from its parent expression by the following rules. The context domain $\mathcal{X}_E$ of the expression E is

- $\mathcal{D}_V \cap \mathcal{D}_E$ in the equation $V = E$.
- $\mathcal{X}_{E'}$ if E' is $\mathrm{op}(\ldots, E, \ldots)$.
- $\mathcal{D}_E \cap \mathcal{X}_{E'}$ when E' is $\mathrm{case} \ldots, E, \ldots \mathrm{esac}$.
- $\mathcal{X}_{E'}$ when E' is $\mathcal{D}' : E$.
- $f(\mathcal{X}_{E'})$ if E' is $E.f$.
- $f_p^{-1}(\mathcal{X}_{E'}) \cap \mathcal{D}_E$ if E' is $\mathrm{reduce}(\oplus, f_p, E)$.

The notion of context domains is important in the automatic simplification of algorithmic complexity [8], since we may have expressions that are defined on a much larger domain than needed. An isolated study of such expressions occurring in the *rhs* of an equation may provide us with an incorrect estimate of the complexity of the equation.

3 Mathematical Background

In this section, we will provide the required mathematical background on linear algebra over integers.

3.1 Matrices

As a convention, we will denote matrices with the upper-case letters and vectors with the lower-case. Unless specifically mentioned, all matrices and vectors have integer elements. We will denote the identity matrix by I.

We will use the following concepts and properties of matrices

- The kernel of a matrix T, written as $\ker(T)$ is the set of all vectors z such that $Tz = 0$.
- A matrix is unimodular if it is square and its determinant is either 1 or -1.
- Two matrices L and L' are said to be *column equivalent* or *right equivalent* if there exists a unimodular matrix U such that $L = L'U$.
- A unique representative element in each set of matrices that are column equivalent is the one in *Hermite normal form* [9].

Definition 1 *An $n \times m$ matrix H with rank d is in* Hermite Normal Form *(HNF) , if*

1. $\forall 1 \leq j \leq d, \exists i_1, \ldots, i_d$ *with* $1 \leq i_1 < \ldots < i_d \leq n$: $H_{i_j,j} > 0$.
2. $\forall 1 \leq j \leq d, 1 \leq i < i_j$: $H_{i,j} = 0$.
3. $\forall d+1 \leq j \leq m, 1 \leq i \leq n$: $H_{i,j} = 0$
4. $\forall 1 \leq l < j \leq d$: $0 \leq H_{i_j,l} < H_{i_j,j}$.

Remark 1 *For every matrix A, there exists a unique matrix H that is in HNF and column equivalent to A* i.e.*, there exists a unimodular matrix U such that $A = HU$.*

Note that the provided definition of the Hermite normal form does not require the matrix A to have full row rank.

There is a related normal form called the *Smith normal form* [18] that we will use in the presentation of this paper.

Definition 2 *An $n \times m$ matrix S with rank d is in* Smith Normal Form *(SNF) , if*

1. *S is a diagonal matrix.*
2. $\forall 1 \leq i \leq d$: $S_{i,i} > 0$.
3. $\forall 1 \leq i \leq d-1$: $S_{i,i}$ *divides* $S_{i+1,i+1}$.
4. $\forall d+1 \leq i \leq \min(n,m)$: $S_{i,i} = 0$

Remark 2 *For every matrix, A, there exists a unique matrix S that is in SNF such that $A = VSU$ where V and U are unimodular matrices.*

3.2 Affine Lattices

The lattice generated by a matrix L is the set of all integer linear combinations of the columns of L. If the columns of a matrix are linearly independent, they constitute a *basis* of the generated lattice. The lattices generated by two matrices are equal *iff* the submatrices corresponding to the non-zero columns in their Hermite normal forms are equal. As a special case, the lattices generated by two $n \times m$ matrices are equal *iff* the matrices are column equivalent.

In this paper, we will use a generalization of the lattices generated by a matrix, additionally allowing offsets by constant vectors. These are called *affine lattices*. An affine lattice is a subset of $\mathbb{Z}^n$ and can be represented as $\{Lz + l | z \in \mathbb{Z}^m\}$ where L and l are an $n \times m$ matrix and n-vector respectively. We call z the coordinates in the particular representation of the affine lattice. Representations of affine lattices will be denoted by $\mathcal{L}$.

The affine lattices represented by $\{Lz + l | z \in \mathbb{Z}^m\}$ and $\{L'z' + l' | z' \in \mathbb{Z}^{m'}\}$ are equal *iff* the matrices generated by L and L' are equal and $l' = Lz_0 + l$ for some constant vector $z_0 \in \mathbb{Z}^m$.

3.3 Integer Polyhedra

An *integer polyhedron*, $\mathcal{P}$ is a subset of $\mathbb{Z}^n$, the elements of which satisfy a finite number of affine inequalities (also called affine constraints or just constraints when there is no ambiguity) with integer coefficients. We follow the convention that the affine constraint c_i is given as $(a_i^T z + \alpha_i \geq 0)$ where $z, a_i \in \mathbb{Z}^n, \alpha_i \in \mathbb{Z}$. The integer polyhedron, $\mathcal{P}$, satisfying the set of constraints $\mathcal{C} = \{c_1, \ldots, c_b\}$ is often written as $\{z \in \mathbb{Z}^n | Qz + q \geq 0\}$ where $Q = (a_1 \ldots a_b)^T$ is an $b \times n$ matrix[2] and $q = (\alpha_1 \ldots \alpha_b)^T$ is an b-vector.

We shall use the following properties and notation.

- The constraint $c \equiv (a^T z + \alpha \geq 0)$ of $\mathcal{P}$ is said to be *saturated iff* $(a^T z + \alpha = 0) \cap \mathcal{P} = \mathcal{P}$.
- The *lineality space* of $\mathcal{P}$ is defined as the linear part of the largest affine subspace contained in $\mathcal{P}$. It is given by $\ker(Q)$.
- The *context* of $\mathcal{P}$ is defined as the linear part of the smallest affine subspace that contains $\mathcal{P}$. If the saturated constraints of $\mathcal{P}$ in $\mathcal{C}$, are the rows of $\{Q_0 z + q_0 \geq 0\}$, then it is $\ker(Q_0)$.

3.4 Affine Images of Integer Polyhedra

Consider the integer polyhedron $\mathcal{P} = \{z \in \mathbb{Z}^m | Qz + q \geq 0\}$ and the affine function $f : (z \rightarrow Tz + t)$ where Q and T are $b \times m$ and $n \times m$ matrices respectively and q and t are a b-vector and n vector respectively. The image of $\mathcal{P}$ under f is of the form $\{Tz + t | Qz + q \geq 0, z \in \mathbb{Z}^m\}$. These are the so called linearly bound lattices (or LBLs) [19].

[2] When Q and/or q is rational, we can appropriately multiply the constraints to get integer elements.

4 The Polyhedral Model

The polyhedral model is a concrete instance of the equational language presented in section 2. As we have already mentioned, the polyhedral model has the family of unions of integer polyhedra as $\mathcal{F}_{\mathcal{D}}$ and the family of affine functions of the form $(z \rightarrow Tz+t)$ as $\mathcal{F}_f$. Recall that all matrices and vectors have integer elements. Since such affine functions are a mapping on z which is the vector of coordinates on the standard basis, we will refer to them as *standard affine functions*. Variables in the polyhedral model may be seen as multi-dimensional arrays.

Our presentation of the language specification in section 2 is based on the Alpha language [14, 10] and the MMAlpha framework for manipulating Alpha programs, which relies on a library for manipulating polyhedra [20].

4.1 Limitations

As mentioned in the introduction, the polyhedral model suffers from the following limitations.

- Loop programs with non-unit stride, and *non-unimodular* transformations fall outside the scope of the model.
- Parallel architectures with *periodic* processor activity, such as *multi-rate arrays* [12] and bidirectional systolic arrays, cannot be described in the model.
- Reduction operations with arbitrary projections cannot be expressed in the model.

Here, we will elaborate the limitation on reductions. This limitation, in essence, arises as a result of image by the family of functions. The family of unions of integer polyhedra is not closed under image by the family of standard affine functions. To account for this shortcoming, projection functions in a reduction have been limited to those valid functions, the image by which, of the particular domain is also a valid domain.

Nevertheless, even with this condition, some problems persisted (not explained before). Recall that the context domain of a (sub)expression in a dependence expression requires the image of a valid domain by the dependence function which is not necessarily a valid domain. This is handled in a ad-hoc manner by taking the closure (convex hull) of the image in $\mathcal{F}_{\mathcal{D}}$.

A key contribution of this paper is the proof that the family of unions of $\mathcal{Z}$-polyhedra is closed under images by the family of functions. The precise characterization of the family of functions will be presented in section 5. Nevertheless, we wish to mention that all standard affine functions are elements of this family.

5 The $\mathcal{Z}$-Polyhedral Model

In this section we will present extensions to the polyhedral model. We will start by studying more general mathematical objects than polyhedra called $\mathcal{Z}$-polyhedra . A $\mathcal{Z}$-polyhedron is the intersection of an integer polyhedron and an affine lattice. When the affine lattice is the canonical lattice, $\mathbb{Z}^n$, the obtained $\mathcal{Z}$-polyhedron is also an integer polyhedron. Since a $\mathcal{Z}$-polyhedron cannot be expressed as a finite union of integer polyhedra, the family of finite unions of $\mathcal{Z}$-polyhedra strictly contains the family of unions of integer polyhedra. Also, as we have previously mentioned the associated family of functions contains the family of standard affine functions. Both these containments are strict, therefore, upon showing the required closure properties, we will have the $\mathcal{Z}$-polyhedral model as a strict generalization of the polyhedral model.

Moreover, we will also show that the family of unions of $\mathcal{Z}$-polyhedra is closed under images by the family of functions. This avoids the irregularities of the polyhedral model, as seen in the previous section.

5.1 Representation of $\mathcal{Z}$-Polyhedra

The key insight into proving the required closure properties on unions of $\mathcal{Z}$-polyhedra is a certain form of representation. We represent the $\mathcal{Z}$-polyhedra in the following form, say $\mathcal{Z}$

$$\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\} \tag{1}$$

where L has full column rank and $\mathcal{P}^c_{\mathcal{Z}} = \{z | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ has a context that is the universe, $\mathbb{Z}^m$. $\mathcal{P}^c_{\mathcal{Z}}$ is called the coordinate polyhedron associated with the particular representation[3], $\mathcal{Z}$, of the $\mathcal{Z}$-polyhedron. The $\mathcal{Z}$-polyhedron for which L has no columns has a coordinate polyhedron in $\mathbb{Z}^0$. The empty $\mathcal{Z}$-polyhedron is denoted by $\{|\}$.

The conditions on L and $\mathcal{P}^c_{\mathcal{Z}}$ in the representation guarantee the following three critical properties

1. Every point in the coordinate polyhedron maps to a unique iteration point of the $\mathcal{Z}$-polyhedron.
2. Two representations with the same affine image, $Lz+l$, are *equivalent iff* their coordinate polyhedra are equal.
3. Two representations, $\{Lz+l|Qz+q \geq 0, z \in \mathbb{Z}^m\}$ and $\{L'z'+l'|Q'z'+q' \geq 0, z' \in \mathbb{Z}^{m'}\}$, are *not equivalent* if the affine lattices represented by $\{Lz+l|z \in \mathbb{Z}^m\}$ and $\{L'z'+l'|z' \in \mathbb{Z}^{m'}\}$ are not equal.

Two representations are said to be equivalent if they correspond to the same set. Naturally, two domains are equal if they have equivalent representations. In the set of properties given above, property 1 and 2 are a consequence of the fact that L has full column rank. Property 3 is a consequence of the fact that the context of $\mathcal{P}^c_{\mathcal{Z}}$ is the universe, $\mathbb{Z}^m$.

We will make extensive use of some the results presented by Le Verge [11] who showed that the family of polyhedra is strictly contained in the family of $\mathcal{Z}$-polyhedra which is turn in strictly contained in the family of LBLs. He proved that membership testing in LBLs is $\mathcal{NP}$-complete. In addition, he proved that it is at least an $\mathcal{NP}$-complete problem to determine if

[3]Note that we have denoted particular representations with the symbol $\mathcal{Z}$. Later, $\mathcal{ZP}$ will denote the set of iterations.

an LBL is a $\mathcal{Z}$-polyhedronand gave sufficient conditions for an LBL to be a $\mathcal{Z}$-polyhedron . We will study these conditions in depth and provide extensions to his result. Finally, he provided a representation of arbitrary LBLs as the canonic projection of an integer polyhedron along a single canonic vector.

Note that $\mathcal{Z}$ is in the form of an affine image of an integer polyhedron (an LBL) and we have mentioned that the family of LBLs is a more general class of objects than the family of $\mathcal{Z}$-polyhedra. However, Le Verge showed that the LBL is a $\mathcal{Z}$-polyhedron when L has full column rank in the context, $\ker(Q_0)$, of $\mathcal{P}^c_{\mathcal{Z}}$. Mathematically, L has full column rank in the context of $\mathcal{P}^c_{\mathcal{Z}}$, *iff* $A = \begin{pmatrix} L \\ Q_0 \end{pmatrix}$ has full column rank. The LBL in (1) is a $\mathcal{Z}$-polyhedron since it satisfies an even stricter condition — L has full column rank. We have imposed such a requirement since under the conditions of Le Verge, properties 2 and 3 given above fails *i.e.*, it is possible for two $\mathcal{Z}$-polyhedra with the same affine image, $Lz + l$, to be equivalent even when their coordinate polyhedra differ and two representations may be equivalent even when the affine lattices in the representation are not equal.

However, our stricter requirement does not mean that we accept a restricted set of objects or limit the expressibility provided to the programmer. We will first show that every $\mathcal{Z}$-polyhedron has a representation that satisfies the condition that L has full column rank. Then, we will present a transformation that converts representations in which L has full column rank but the coordinate polyhedron does not have the entire universe as its context to the required form presented in (1).

We will then extend the result of Le Verge by weakening his conditions even further; instead of requiring A to have full column rank which implies that $\ker(A) = \{0\}$, we will require $\ker(A) \subseteq \ker(Q)$. Thus, we provide the programmer with even greater expressivity than was available previously. That LBLs satisfying our weaker conditions are also $\mathcal{Z}$-polyhedra will be shown by providing a transformation that converts representations satisfying $\ker(A) \subseteq \ker(Q)$ to equivalent representations satisfying $\ker(A) = \{0\}$. Finally, we will provide a transformation that converts these representations satisfying $\ker(A) = \{0\}$ to the form satisfying $\ker(L) = \{0\}$ that can be brought to the final required form by the transformation mentioned previously.

Since it is at least an $\mathcal{NP}$-complete problem to determine if an LBL is a $\mathcal{Z}$-polyhedron and all the conditions presented above take polynomial time to verify, these are just *sufficient* conditions.

5.1.1 Completeness of the Representation

Here we will show that every $\mathcal{Z}$-polyhedron can be represented in the required form presented in (1). Consider the $\mathcal{Z}$-polyhedron, $\mathcal{ZP}$ say, which is the intersection of the integer polyhedron $\mathcal{P} = \{y \in \mathbb{Z}^n | Qy + q \geq 0\}$ and the affine lattice represented by $\mathcal{L} = \{Lz + l | z \in \mathbb{Z}^m\}$ where Q and L are $b \times n$ and $n \times m$ matrices respectively and q and l are an b-vector and a n-vector respectively. Note that both $\mathcal{P}$ and the affine lattice represented by $\mathcal{L}$ lie in $\mathbb{Z}^n$.

We will show the completeness of the representation in two steps, first by showing that any $\mathcal{Z}$-polyhedron can be represented in a form where L has full column rank. Then, we will transform this representation to the required form.

Step 1: Let L have rank d and $H = \begin{pmatrix} H' & 0 \end{pmatrix}$ be its Hermite normal form such that H' is a $n \times d$ matrix of full column rank and $L = HU$ where U is a unimodular matrix. The representation $\mathcal{L}$ can be written as $\{HUz + l | z \in \mathbb{Z}^m\}$. Since U is unimodular, $\mathcal{L}$ is equivalent to $\{Hz' + l | z' \in \mathbb{Z}^m\}$ where $z' = Uz$.

Now consider $\mathcal{ZP}$, the intersection of the affine lattice represented by $\mathcal{L}$ and the polyhedron $\mathcal{P}$.

$$\begin{aligned} \mathcal{ZP} &= \{Hz' + l | z' \in \mathbb{Z}^m\} \cap \{y \in \mathbb{Z}^n | Qy + q \geq 0\} \\ &= \{Hz' + l | Q(Hz' + l) + q \geq 0, z' \in \mathbb{Z}^m\} \\ &= \{\begin{pmatrix} H' & 0 \end{pmatrix} z' + l | Q \begin{pmatrix} H' & 0 \end{pmatrix} z' + \\ &\qquad\qquad (q + Ql) \geq 0, z' \in \mathbb{Z}^m\} \\ &= \{H'z'' + l | Q'z'' + q' \geq 0, z'' \in \mathbb{Z}^d\} \end{aligned}$$

where $Q' = QH'$, $q' = q + Ql$ and $z'' = \begin{pmatrix} I & 0 \end{pmatrix} z'$. Since, H' has full column rank, the representation can be brought to the final required form using the following transformation.

Step 2: To show the completeness of our representation scheme, we need to prove that every representation of the form

$$\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\} \tag{2}$$

satisfying $\ker(L) = \{0\}$ can be transformed to the final required form.

Let $\mathcal{P}^c_{\mathcal{Z}} = \{z | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ be the coordinate polyhedron associated to the representation. Let the smallest affine subspace that contains $\mathcal{P}^c_{\mathcal{Z}}$ be $\{z | Tz = t\}$ where T is an $n \times m$ matrix and t is an n-vector. The context of $\mathcal{P}^c_{\mathcal{Z}}$ is given by $\ker(T)$. Let T have rank d and let S be its Smith normal form such that $T = VSU$, where V and U are unimodular matrices. We have $VSUz = t$. Since, U is unimodular, we may change the coordinates to $z' = Uz$ to get the following equivalent representation of the $\mathcal{Z}$-polyhedron

$$\{L'z' + l | Q'z' + q \geq 0, z' \in \mathbb{Z}^m\} \tag{3}$$

where $L' = LU^{-1}$ and $Q' = QU^{-1}$.

We have $Sz' = t'$ where $t' = V^{-1}t$. If any of the $(d+1)^{\text{th}}, \ldots, n^{\text{th}}$ component of t' is non-zero then the coordinate polyhedron is empty implying the corresponding $\mathcal{Z}$-polyhedron is empty. If S' is the top-left $d \times d$ submatrix of S, t'' is the vector constructed from the first d elements of t' and $z' = \begin{pmatrix} z_1 \\ z'' \end{pmatrix}$, we have $\begin{pmatrix} S' & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z'' \end{pmatrix} = \begin{pmatrix} t'' \\ 0 \end{pmatrix}$ or

$$\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z'' \end{pmatrix} = \begin{pmatrix} (S')^{-1}t'' \\ 0 \end{pmatrix}$$

These give us d equalities for the first d elements of z'. Again, if any elements of the vector $(S')^{-1}t''$ are rational, then the coordinate polyhedron is empty and therefore the corresponding $\mathcal{Z}$-polyhedron is empty. Otherwise, substituting $\begin{pmatrix} (S')^{-1}t'' \\ z'' \end{pmatrix}$ for z' in (3), we get the following equivalent representation

$$\{L''z'' + l' | Q''z'' + q' \geq 0, z'' \in \mathbb{Z}^{m-d}\} \tag{4}$$

where $L' = \begin{pmatrix} L_1 & L'' \end{pmatrix}$, $l' = l + L_1(S')^{-1}t''$, $Q' = \begin{pmatrix} Q_1 & Q'' \end{pmatrix}$ and $q' = q + Q_1(S')^{-1}t''$. Note, the T was of rank d and therefore $\mathcal{P}^c_{\mathcal{Z}}$ had to be a $m-d$ dimensional polyhedron embedded in $\mathbb{Z}^m$. Recall that L has full column rank and therefore the iteration domain has a one-to-one mapping with points in the coordinate domain in (2). The coordinate polyhedron in (4) is embedded in $\mathbb{Z}^{m-d}$. Therefore, it necessarily must have a context that is the entire universe.

5.1.2 Expressivity

Here, we will present the expressivity available to the programmer and the transformations needed to convert user specifications to the required form.

Relaxation of the Sufficient Condition Le Verge proved that an LBL of the form $\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ is a $\mathcal{Z}$-polyhedron when $\ker(A) = \{0\}$ where $A = \begin{pmatrix} L \\ Q_0 \end{pmatrix}$ and $\ker(Q_0)$ is the context of the polyhedron $\{z | Qz + q \geq 0, z \in \mathbb{Z}^m\}$. We will present a relaxation of this sufficient condition, only requiring $\ker(A) \subseteq \ker(Q)$, by providing a transformation that converts such representations to equivalent representations satisfying $\ker(A) = \{0\}$. In addition to providing greater expressivity to the programmer, this theorem is crucial to prove closure of unions of $\mathcal{Z}$-polyhedra under image.

We will use the results of Le Verge's proposition 4.4 [11]. It is repeated here for convenience.

Proposition 1 *Let M be an integral matrix. The following properties are equivalent:*

1. *there exists an integral matrix M' such that $M'M = I$;*
2. *there exists an integral matrix N such that $\begin{pmatrix} M & N \end{pmatrix}$ is unimodular;*
3. *The Hermite normal form of M^T is $\begin{pmatrix} I & 0 \end{pmatrix}$*

The following theorem claims the equivalence of LBLs resulting from our more general representations to those satisfying Le Verge's sufficient condition.

Theorem 1 *A representation $\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ of an LBL satisfying ker$(A) \subseteq$ ker(Q), where $A = \begin{pmatrix} L \\ Q_0 \end{pmatrix}$ and the context of its coordinate polyhedron given by ker(Q_0), can be transformed to an equivalent representation of the form $\{L'z' + l | Q'z' + q \geq 0, z' \in \mathbb{Z}^{m'}\}$ where L' has full column rank in the context of its coordinate polyhedron.*

Proof Let the A be of rank d and $H = \begin{pmatrix} H' & 0 \end{pmatrix}$ be its Hermite normal form such that H' is the submatrix of full column rank corresponding to the first d columns of H. From definition, there exists a unimodular matrix U such that $A = HU$. We have $AU^{-1} = \begin{pmatrix} H' & 0 \end{pmatrix}$ where U^{-1} is the unimodular inverse of U. Let $U = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ and $U^{-1} = \begin{pmatrix} V_1' & V_2' \end{pmatrix}$ where V_2' is the column submatrix of U^{-1} corresponding to the zero-columns of $\begin{pmatrix} H' & 0 \end{pmatrix}$, $V_1V_1' = I$, $V_1V_2' = 0$, $V_2V_1' = 0$ and $V_2V_2' = I$. Let us construct a matrix, W such that $W \begin{pmatrix} V_1' & V_2' \end{pmatrix} = \begin{pmatrix} V_1' & 0 \end{pmatrix}$ or $W = \begin{pmatrix} V_1' & 0 \end{pmatrix} U$ or $W = V_1'V_1$.

Since, $\ker(A) \subseteq \ker(L)$, $LV_2' = 0$ and therefore $Lz + l = LWz + l$. Also since, $\ker(A) \subseteq \ker(Q)$ we get $Qz + q = QWz + q$. With this, the representation $\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ is equivalent to $\{LWz + l | QWz + q \geq 0, z \in \mathbb{Z}^m\}$ or

$$\{LV_1'V_1z + l | QV_1'V_1z + q \geq 0, z \in \mathbb{Z}^m\} \tag{5}$$

The Hermite normal form for V_1 is $\begin{pmatrix} I & 0 \end{pmatrix}$ by proposition 1 since $V_1V_1' = V_1'^TV_1^T = I$. For unimodular matrix U', we have

$$\begin{aligned} \{V_1z | z \in \mathbb{Z}^m\} &= \{\begin{pmatrix} I & 0 \end{pmatrix} U'z | z \in \mathbb{Z}^m\} \\ &= \{\begin{pmatrix} I & 0 \end{pmatrix} z' | z' \in \mathbb{Z}^m\} \\ &= \{z'' | z'' \in \mathbb{Z}^d\} \end{aligned}$$

Denoting V_1z by z'' in (5), we get the following equivalent representation

$$\{LV_1'z'' + l | QV_1'z'' + q \geq 0, z'' \in \mathbb{Z}^d\} \tag{6}$$

We will now show that (6) is the transformed equivalent representation where LV_1' has full column rank in the context, say $\ker(Q_0')$, of its coordinate polyhedron $\{z'' | QV_1'z'' + q \geq 0, z'' \in \mathbb{Z}^d\}$. Note, $\ker(Q_0V_1')$ is a superset $\ker(Q_0')$.

$$\begin{aligned} \ker \begin{pmatrix} LV_1' \\ Q_0' \end{pmatrix} &\subseteq \ker \begin{pmatrix} LV_1' \\ Q_0V_1' \end{pmatrix} \\ &= \ker \left(\begin{pmatrix} L \\ Q_0 \end{pmatrix} V_1' \right) \\ &= \ker(AV_1') \\ &= \ker(H') \\ &= \{0\} \end{aligned}$$

since H' has full column rank. ∎

Although Le Verge has already been proved that representations satisfying $\ker(A) = \{0\}$ are $\mathcal{Z}$-polyhedra which can be expressed in the required form presented in (1), for the sake of completeness, we present the following transformation that converts these representations to an equivalent representation satisfying $\ker(L) = \{0\}$ which can be brought to the final required form using the transformation in step 2 presented in section 5.1.1.

Consider a representation, $\mathcal{Z}$, of the form $\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ satisfying $\ker(A) = \{0\}$ where $A = \begin{pmatrix} L \\ Q_0 \end{pmatrix}$ and $\ker(Q_0)$ is the context of the polyhedron $\mathcal{P}^c_{\mathcal{Z}} = \{z | Qz + q \geq 0, z \in \mathbb{Z}^m\}$.

Le Verge showed in [11] that the condition $\ker(A) = \{0\}$ holds *iff* there exists an affine function, $f'(y) = Ky + k$, *with rational elements* such that $f'(f(z)) = z$ for all $z \in \mathcal{P}^c_{\mathcal{Z}}$ where $f(z) = Lz + l$. With the presence of such a "restricted inverse", the $\mathcal{Z}$-polyhedron can be equivalently represented as $\{Lz + l | Q(K(Lz + l) + k) + q \geq 0, z \in \mathbb{Z}^m\}$. Simplifying the representation, we get the form

$$\{Lz + l | QKLz + (q + QKl + Qk) \geq 0, z \in \mathbb{Z}^m\} \tag{7}$$

Since, K and k are rational, we may need to multiply the constraints of the coordinate polyhedron by appropriate constants to get integer elements for all matrices and vectors. We will hereafter assume that such a transformation has been performed.

Let L have rank d and $H = \begin{pmatrix} H' & 0 \end{pmatrix}$ be its Hermite normal form such that H' is a $n \times d$ matrix of full column rank and $L = HU$ where U is a unimodular matrix. The representation in (7) is equivalent to

$$\{HUz + l | Q'HUz + q' \geq 0, z \in \mathbb{Z}^m\}$$

where $Q' = QK$ and $q' = q + QKl + Qk$. Since U is unimodular, we have an equivalent representation given by

$$\begin{aligned} &\{Hz' + l | Q'Hz' + q' \geq 0, z' \in \mathbb{Z}^m\} \\ &= \{\begin{pmatrix} H' & 0 \end{pmatrix} z' + l | Q' \begin{pmatrix} H' & 0 \end{pmatrix} z' + q' \geq 0, z' \in \mathbb{Z}^m\} \end{aligned}$$

where $z' = Uz$. Finally by changing to coordinates $z'' = \begin{pmatrix} I & 0 \end{pmatrix} z'$, we get

$$\{H'z'' + l | Q''z'' + q' \geq 0, z'' \in \mathbb{Z}^d\}$$

where $Q'' = Q'H'$. Since, H' has full column rank, the representation can be brought to the final required form using the transformation in step 2 presented in section 5.1.1.

Henceforth we will assume that all representations of $\mathcal{Z}$-polyhedra conform to our representation scheme presented in (1).

5.1.3 Interpretation

The conventional way to interpret our representation of $\mathcal{Z}$-polyhedra as $\{Lz+l | Qz+q \geq 0, z \in \mathbb{Z}^m\}$ is similar to the definition of an LBL, as an affine image of an integer polyhedron. We wish to motivate an alternate view in which $\{Lz + l | z \in \mathbb{Z}^m\}$ in the $\mathcal{Z}$-polyhedral representation is interpreted as a representation of an affine lattice. The $\mathcal{Z}$-polyhedral representation is said to be *based on* the representation of the affine lattice. The set of valid coordinates is given by the coordinate polyhedron. Iteration points of the $\mathcal{Z}$-polyhedral domain are points of the affine lattice in the particular representation corresponding to valid coordinates.

5.1.4 Equivalence

Our representation for $\mathcal{Z}$-polyhedra is such that any two $\mathcal{Z}$-polyhedral representations based on the same representation of an affine lattice are equivalent *iff* their corresponding coordinate polyhedra are equal. We will now study the equivalence of $\mathcal{Z}$-polyhedral representations based on different representations of the same affine lattice. Recall that in our representation scheme, $\mathcal{Z}$-polyhedral representations on different affine lattices are necessarily different. Consider the representation of a $\mathcal{Z}$-polyhedron

$$\{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\} \tag{8}$$

Let $\{L'z' + l' | z' \in \mathbb{Z}^m\}$ be a different representation of the affine lattice in the $\mathcal{Z}$-polyhedral representation. Note that as a consequence of our representation scheme, L' necessarily has the same number of columns as L. By definition $L' = LU$ and $l' = Lz_0 + l$ for some constant vector $z_0 \in \mathbb{Z}^m$. The relationship between the coordinates in the two lattices is simply

$$\begin{aligned} Lz + l &= L'z' + l' \\ &= LUz' + Lz_0 + l \\ &= L(Uz' + z_0) + l \end{aligned}$$

Since L has full column rank, we have $z = Uz' + z_0$. Substituting for z in (8), we get the following equivalent representation

$$\{L'z' + l' | Q(Uz' + z_0) + q \geq 0, z' \in \mathbb{Z}^m\}$$

With this characterization, we have precisely decomposed the problem of equivalence of $\mathcal{Z}$-polyhedral representations to the problem of equivalence of representations of affine lattices and the equality of polyhedra. Our equivalence is precise in the sense that if two $\mathcal{Z}$-polyhedra differ we will be able to provide an iteration point in their difference, otherwise, we will guarantee equality. This is a direct consequence of our representation scheme. In previous works [16], the representation of $\mathcal{Z}$-polyhedra was such that, in some cases, the equivalence of two representations could not be guaranteed even when the domains were identical.

5.1.5 Canonical Form

The representation of a $\mathcal{Z}$-polyhedron is not unique. Here, we will present a canonical form for the representation of $\mathcal{Z}$-polyhedra. For this, we present a canonical form for the representation of affine lattices.

Definition 3 *An affine lattice of the form* $\{Lz + l | z \in \mathbb{Z}^m\}$ *where the* $n \times m$ *matrix* L *has full column rank is in canonical form if*

1. *L is in Hermite normal form.*
2. *$\forall 1 \leq j \leq m$: $l_{i_j} < L_{i_j,j}$ where $L_{i_j,j}$ is the first non-zero element in column j.*

As a property of our representation scheme, two $\mathcal{Z}$-polyhedral representations based on the same lattice are equivalent *iff* their coordinate polyhedra are equal. Now, if we choose any previously used canonical form for the representation of the coordinate polyhedra, we have a canonical representation for $\mathcal{Z}$-polyhedra.

5.2 Unions of $\mathcal{Z}$-Polyhedra

Domains in the $\mathcal{Z}$-polyhedral model are finite unions of $\mathcal{Z}$-polyhedra where each element in the union is expressed in the representation discussed in section 5.1. To be an instance of the equational language, unions of $\mathcal{Z}$-polyhedra must be closed under intersection, finite union and difference and image and preimage by the family of functions. Here we will show closure under the intersection and difference (The union of a finite set of unions of $\mathcal{Z}$-polyhedra is trivially a finite set of $\mathcal{Z}$-polyhedra).

In section 5.3 we will define the family of functions and then demonstrate closure of the family of unions of $\mathcal{Z}$-polyhedra under image and preimage by the family of functions.

5.2.1 Intersection

From elementary set theory, the intersection of two unions of $\mathcal{Z}$-polyhedra , given as $\mathcal{D} = \bigcup_i \mathcal{ZP}_i$ and $\mathcal{D}' = \bigcup_j \mathcal{ZP}'_j$, equals the union of intersections of two $\mathcal{Z}$-polyhedra as follows

$$\mathcal{D} \cap \mathcal{D}' = (\bigcup_i \mathcal{ZP}_i) \cap (\bigcup_j \mathcal{ZP}'_j) = \bigcup_{i,j} (\mathcal{ZP}_i \cap \mathcal{ZP}'_j)$$

Thus, we only need to show that the intersection of two $\mathcal{Z}$-polyhedra is a finite union of $\mathcal{Z}$-polyhedra . As a matter of fact, it is precisely a single $\mathcal{Z}$-polyhedron . Let the two $\mathcal{Z}$-polyhedra be represented by, say $\mathcal{Z}$ and $\mathcal{Z}'$ as follows

$$\begin{aligned} \mathcal{Z} &= \{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\} \\ \mathcal{Z}' &= \{L'z' + l' | Q'z' + q' \geq 0, z' \in \mathbb{Z}^{m'}\} \end{aligned}$$

The intersection of $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ and $\mathcal{Z}'$ relies on the intersection of the affine lattices represented by $\mathcal{L} = \{Lz + l | z \in \mathbb{Z}^m\}$ and $\mathcal{L}' = \{L'z' + l' | z' \in \mathbb{Z}^{m'}\}$ on which they are based. The intersection of affine lattices represented by $\mathcal{L}$ and $\mathcal{L}'$ is an affine lattice, say represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$ where L'' has full column rank. The affine lattice represented by $\mathcal{L}''$ may be empty, in which case the corresponding $\mathcal{Z}$-polyhedron is empty. If $\mathcal{L}''$ represents a non-empty affine lattice, we have the following relationships. Note, L'' may have a fewer number of columns than either L or L'.

$$\begin{aligned} L'' = LS, l'' = Ls + l \\ L'' = L'S', l'' = L's' + l' \end{aligned}$$

where S and S' are matrices and s and s' are vectors.

Taking the intersection of the $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ and $\mathcal{Z}'$ with the affine lattice represented by $\mathcal{L}''$ we get

$$\begin{aligned}\mathcal{Z} \cap \mathcal{L}'' &= \{L''z'' + l'' | Q(Sz'' + s) + q \geq 0, z'' \in \mathbb{Z}^{m''}\} \\ \mathcal{Z}' \cap \mathcal{L}'' &= \{L''z'' + l'' | Q'(S'z'' + s') + q' \geq 0, z'' \in \mathbb{Z}^{m''}\}\end{aligned}$$

Since, they are based on the same representation of the affine lattice, the intersection of these two $\mathcal{Z}$-polyhedra is simply

$$\{L''z'' + l'' | \begin{pmatrix} QS \\ Q'S' \end{pmatrix} z'' + \begin{pmatrix} q + Qs \\ q' + Q'S' \end{pmatrix} \geq 0, z'' \in \mathbb{Z}^{m''}\}$$

Note, the coordinate polyhedra of this intersection may not have the entire universe as its context, in which case, we would bring it to the required representation through the transformation in step 2 presented in section 5.1.1.

5.2.2 Difference

From set theory, the difference of two unions of $\mathcal{Z}$-polyhedra, given as $\mathcal{D} = \bigcup_i \mathcal{ZP}_i$ and $\mathcal{D}' = \bigcup_j \mathcal{ZP}'_j$, equals the union of differences of two $\mathcal{Z}$-polyhedra as follows

$$\mathcal{D} - \mathcal{D}' = (\bigcup_i \mathcal{ZP}_i) - (\bigcup_j \mathcal{ZP}'_j) = \bigcup_i \left(\bigcap_j (\mathcal{ZP}_i - \mathcal{ZP}'_j) \right)$$

If we show that the difference of two $\mathcal{Z}$-polyhedra is a finite union of $\mathcal{Z}$-polyhedra , we may use the result on closure of domains under intersection presented in the previous section and claim closure under difference.

Let the two $\mathcal{Z}$-polyhedra be represented by, say $\mathcal{Z}$ and $\mathcal{Z}'$ as follows

$$\begin{aligned}\mathcal{Z} &= \{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\} \\ \mathcal{Z}' &= \{L'z' + l' | Q'z' + q' \geq 0, z' \in \mathbb{Z}^{m'}\}\end{aligned}$$

The difference of $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ and $\mathcal{Z}'$ relies on the intersection and difference of the affine lattices represented by $\mathcal{L} = \{Lz + l | z \in \mathbb{Z}^m\}$ and $\mathcal{L}' = \{L'z' + l' | z' \in \mathbb{Z}^{m'}\}$ on which they are based. The difference of $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ and $\mathcal{Z}'$ is a union of $\mathcal{Z}$-polyhedra elements of which may be defined on either the intersection or the difference of the affine lattices.

Let us first consider the intersection of the affine lattices. The intersection of affine lattices represented by $\mathcal{L}$ and $\mathcal{L}'$ is an affine lattice, say represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$ where L'' has full column rank. The affine lattice represented by $\mathcal{L}''$ may be empty, in which case all $\mathcal{Z}$-polyhedra corresponding to the intersection are empty. If $\mathcal{L}''$ represents a non-empty affine lattice, we have the following relationships.

$$\begin{aligned}&L'' = LS, l'' = Ls + l \\ &L'' = L'S', l'' = L's' + l'\end{aligned}$$

where S and S' are matrices and s and s' are vectors.

Taking the intersection of the $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ and $\mathcal{Z}'$ with the affine lattice represented by $\mathcal{L}''$ we get

$$\begin{aligned} \mathcal{Z} \cap \mathcal{L}'' &= \{L''z'' + l'' | Q(Sz'' + s) + q \geq 0, z'' \in \mathbb{Z}^{m''}\} \\ \mathcal{Z}' \cap \mathcal{L}'' &= \{L''z'' + l'' | Q'(S'z'' + s') + q' \geq 0, z'' \in \mathbb{Z}^{m''}\} \end{aligned}$$

Since, they are based on the same representation of the affine lattice, the difference of these two $\mathcal{Z}$-polyhedra is simply a union of $\mathcal{Z}$-polyhedra , each element of which (indexed by k) can be represented by

$$\{L''z'' + l'' | Q''_k z'' + q''_k \geq 0, z'' \in \mathbb{Z}^{m''}\}$$

where the following difference of polyhedra $\{z'' | Q(Sz''+s)+q \geq 0, z'' \in \mathbb{Z}^{m''}\} - \{z'' | Q'(S'z''+s')+q' \geq 0, z'' \in \mathbb{Z}^{m''}\}$ is a union of polyhedra of the form $\{z'' | Q''_k z'' + q''_k \geq 0, z'' \in \mathbb{Z}^{m''}\}$. Let the obtained union of $\mathcal{Z}$-polyhedra be denoted by $\mathcal{D}_{\text{int}}$. The coordinate polyhedra of elements in this union may not have the entire universe as its context, in which case, we would bring them to the required representation through the transformation in step 2 presented in section 5.1.1.

Now, let us consider the difference of the affine lattices. The difference of affine lattices represented by $\mathcal{L}$ and $\mathcal{L}'$ is a union of non-empty affine lattices (indexed by h), say represented by $\mathcal{L}^{\#}_h = \{L^{\#}_h z^{\#}_h + l^{\#}_h | z^{\#}_h \in \mathbb{Z}^{m^{\#}_h}\}$ where each $L^{\#}_h$ has full column rank. We have the following relationships.

$$L^{\#}_h = LS_h, l^{\#}_h = Ls_h + l$$

where S_h is an matrix and s_h is an vector. The intersection of the $\mathcal{Z}$-polyhedra represented by $\mathcal{Z}$ with the affine lattices represented by $\mathcal{L}^{\#}_h$ can be represented by

$$\mathcal{Z} \cap \mathcal{L}^{\#}_h = \{L^{\#}_h z^{\#}_h + l^{\#}_h | QS_h z^{\#}_h + (q + Qs_h) \geq 0, z^{\#}_h \in \mathbb{Z}^{m^{\#}_h}\}$$

Let this union be denoted by $\mathcal{D}_{\text{diff}}$. The coordinate polyhedra of elements in this union may not have the entire universe as its context, in which case, we would bring them to the required representation through the transformation in step 2 presented in section 5.1.1.

5.3 Affine Functions on a Lattice

We will now define the family of functions defined on unions of $\mathcal{Z}$-polyhedra. We allow functions of the form $(Kz + k \rightarrow Rz + r)$, where K has full column rank. Such functions provide a mapping from the iteration $Kz + k$ to the iteration $Rz + r$. We will call these *affine functions on a lattice* or *affine lattice functions*. We have imposed that K has full column rank to guarantee that the function maps a single point in its domain to a single point in its range.

5.3.1 Preimage

The preimage of a union of $\mathcal{Z}$-polyhedra is the union of the preimage of individual $\mathcal{Z}$-polyhedra. We therefore only need to show that the preimage of a single $\mathcal{Z}$-polyhedron, represented by say $\mathcal{Z}$, is a finite union of $\mathcal{Z}$-polyhedra . As a matter of fact, it is precisely a single $\mathcal{Z}$-polyhedron . Let the representation $\mathcal{Z}$ be

$$\mathcal{Z} = \{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$$

Let the desired preimage on $\mathcal{Z}$ be by the function represented as $(Kz' + k \rightarrow Rz' + r)$ where K has full column rank, m' say. By definition, the function provides a mapping from the iteration point $Kz' + k$ to the iteration point $Rz' + r$. Since, we are concerned with the preimage, if the iteration point $Rz' + r$ lies in the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ the iteration point in the preimage is $Kz' + k$. However, $Rz' + r$ may not necessarily lie in the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ for all values of z'. Specifically, a preimage exists only for iteration points in the intersection the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ and the affine lattice represented by $\{Rz' + r | z' \in \mathcal{Z}^{m'}\}$. Consider the intersection of the affine lattices represented by $\mathcal{L} = \{Lz + l | z \in \mathbb{Z}^m\}$ and $\mathcal{L}' = \{Rz' + r | z' \in \mathcal{Z}^{m'}\}$. The intersection of affine lattices represented by $\mathcal{L}$ and $\mathcal{L}'$ is an affine lattice, say represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$ where L'' has full column rank. The affine lattice represented by $\mathcal{L}''$ may be empty, in which case the preimage is also empty. If $\mathcal{L}''$ represents a non-empty affine lattice, we have the following relationships.

$$\begin{array}{c} L'' = LS, l'' = Ls + l \\ L'' = RS', l'' = Rs' + r \end{array}$$

where S and S' are matrices and s and s' are vectors.

Taking the intersection of the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ and the affine lattice represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$, we get the following $\mathcal{Z}$-polyhedron.

$$\mathcal{Z} \cap \mathcal{L}'' = \{L''z'' + l'' | QSz'' + (q + Qs) \geq 0, z'' \in \mathbb{Z}^{m''}\} \tag{9}$$

A preimage by the function exists only for iterations in the $\mathcal{Z}$-polyhedron given above. Therefore, we may safely restrict the function to map to iteration points in the lattice $\{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$. We will first characterize values of z' for which $Rz' + r$ lies in the lattice $\{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$. Substituting for L'' and l'', we have $L''z'' + l'' = RS'z'' + Rs' + r = R(S'z'' + s') + r$. This equals $Rz' + r$ to get the desired values of z'. Let R be of rank d and $H = \begin{pmatrix} H' & 0 \end{pmatrix}$ be its Hermite normal form such that $R = HU$. Substituting for R we get

$$\begin{aligned} \begin{pmatrix} H' & 0 \end{pmatrix} U(S'z'' + s') &= \begin{pmatrix} H' & 0 \end{pmatrix} Uz' \\ \begin{pmatrix} H' & 0 \end{pmatrix} S''z'' + s'' &= \begin{pmatrix} H' & 0 \end{pmatrix} Uz' \end{aligned}$$

where $S'' = US'$ and $s'' = \begin{pmatrix} H' & 0 \end{pmatrix} Us'$. Since, U is a unimodular matrix, we will replace Uz' by $z^{\#}$ to get $(KU^{-1}z^{\#} + k \rightarrow Hz^{\#} + r)$ as an equivalent representation of the function by which a preimage is desired and $\begin{pmatrix} H' & 0 \end{pmatrix} S''z'' + s'' = \begin{pmatrix} H' & 0 \end{pmatrix} z^{\#}$. If $z^{\#} = \begin{pmatrix} z_1^{\#} \\ z_2^{\#} \end{pmatrix}$,

$S'' = \begin{pmatrix} S''_1 \\ S''_2 \end{pmatrix}$ and $s'' = \begin{pmatrix} s''_1 \\ s''_2 \end{pmatrix}$ then we have $z^\#_1 = S''_1 z'' + s''_1$. As explained, we may safely restrict the function to map to the iteration points in the lattice $\{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$. Substituting for $z^\#$ and $z^\#_1$, we get the following representation for the function

$$KU^{-1} \begin{pmatrix} S''_1 z'' + s''_1 \\ z^\#_2 \end{pmatrix} + k \rightarrow L''z'' + l''$$

If $KU^{-1} = \begin{pmatrix} K'_1 & K'_2 \end{pmatrix}$ we have the following representation of the function. $K'_1 S''_1 z'' + K'_1 s''_1 + K'_2 z^\#_2 + k \rightarrow L''z'' + l''$ which is equivalent to

$$\begin{pmatrix} K'_1 S''_1 & K'_2 \end{pmatrix} \begin{pmatrix} z'' \\ z^\#_2 \end{pmatrix} + (K'_1 s''_1 + k) \rightarrow L''z'' + l''$$

Note, that $z^\#_2$ is unconstrained. We may now represent the preimage by

$$\begin{aligned} \Big\{ & \begin{pmatrix} K'_1 S''_1 & K'_2 \end{pmatrix} \begin{pmatrix} z'' \\ z^\#_2 \end{pmatrix} + (K'_1 s''_1 + k) \\ & | \begin{pmatrix} QS & 0 \end{pmatrix} \begin{pmatrix} z'' \\ z^\#_2 \end{pmatrix} + (q + Qs) \geq 0 \\ & , \begin{pmatrix} z'' \\ z^\#_2 \end{pmatrix} \in \mathbb{Z}^{m'-d+m''} \Big\} \end{aligned} \qquad (10)$$

Since, $\begin{pmatrix} K'_1 & K'_2 \end{pmatrix}$ has full column rank, $\begin{pmatrix} K'_1 S''_1 & K'_2 \end{pmatrix}$ will have full column rank *iff* $K'_1 S''_1$ has full column rank. Note that $L'' = \begin{pmatrix} H' & 0 \end{pmatrix} \begin{pmatrix} S''_1 \\ S''_2 \end{pmatrix} = H' S''_1$ where both L'' and H' have full column rank. Therefore, S''_1 necessarily must have full row rank, which in turn implies that $K'_1 S''_1$ has full column rank. The coordinate polyhedra for (10) may not have the entire universe as its context, in which case, we would bring it to the required representation through the transformation in step 2 presented in section 5.1.1.

Discussion We wish to mention that it is necessary to consider the general case where the affine lattice in the range of a function may not necessarily be a subset of the affine lattice on which the $\mathcal{Z}$-polyhedron is based. If otherwise, we had restricted the function to have its range defined over the appropriate affine sublattices (which would have significantly simplified the calculations presented above), the semantic equivalence of $\mathcal{Z}$-polyhedra based solely on the set of iteration points, would fail. For example, the function $(i \rightarrow i)$ would be defined on the $\mathcal{Z}$-polyhedron represented by $\{i | i \geq 0, i \in \mathbb{Z}\}$ but would not be defined on the equivalent union of two $\mathcal{Z}$-polyhedra represented by $\{2i | i \geq 0, i \in \mathbb{Z}\}$ and $\{2i+1 | i \geq 0, i \in \mathbb{Z}\}$.

5.3.2 Change of Basis

Before, we talk about arbitrary images of unions of $\mathcal{Z}$-polyhedra by affine functions on lattices, we will study a restricted case in which

1. each element[4] in the union is of the form

$$\{L_i z_i + l_i | Q_i z_i + q_i \geq 0, z_i \in \mathbb{Z}^m\}$$

2. the affine lattice function represented as $(Kz' + k \rightarrow Rz' + r)$ is such that K as well as R have full dimensional column rank, m.

3. The affine lattices $\{L_i z_i + l_i | z_i \in \mathbb{Z}^m\}$ related to each $\mathcal{Z}$-polyhedronin the union are sublattices of the affine lattice $\{Kz' + k | z' \in \mathbb{Z}^m\}$ in the domain of the affine lattice function.

The change of basis is a frequently used space reindexing function to perform semantically equivalent transformations of the specification. We will first study the image of each $\mathcal{Z}$-polyhedronby the change of basis function. Then, we will discuss the transformation of the specification.

By property 3 above,

$$L_i = KS_i, l_i = Ks_i + k$$

where S_i are matrices and s_i are vectors. We may safely restrict the change of basis on a $\mathcal{Z}$-polyhedron to be defined over the affine sublattice related to the $\mathcal{Z}$-polyhedron. The restricted change of basis is

$$(K(S_i z_i + s_i) + k \rightarrow R(S_i z_i + s_i) + r)$$

which is equivalent to

$$(L_i z_i + l_i \rightarrow RS_i z_i + (r + Rs_i))$$

The image can be represented as

$$\{RS_i z_i + (r + Rs_i) | Q_i z_i + q_i \geq 0, z_i \in \mathbb{Z}^m\}$$

Note, RS_i has full column rank since R has full column rank and S_i has full row rank. Also the coordinate polyhedron is identical to the original $\mathcal{Z}$-polyhedron, and so has the entire universe as its context.

The transformation of the specification under a change of basis, f of the variable X is the following.

1. Replace the domain of X by its image under f.

2. Replace all occurrences of X on the *rhs* of any equation by $X.f$

3. For the equation defining X, add a dependence by f^{-1} on the expression on its *rhs*.

[4]In some cases, we may even choose to waive the condition on the representation of $\mathcal{Z}$-polyhedra that the coordinate polyhedron has the entire universe as its context.

5.3.3 Images

We will now discuss images of unions of $\mathcal{Z}$-polyhedra by arbitrary affine functions on lattices. The image of a union of $\mathcal{Z}$-polyhedra by a function is the union of images of individual $\mathcal{Z}$-polyhedra .

We already know that the image of a $\mathcal{Z}$-polyhedron by an arbitrary affine function is an LBL which is a more general class of objects than $\mathcal{Z}$-polyhedra. Here, we will prove a surprising result that any LBL is a union of $\mathcal{Z}$-polyhedra. Thus, the family of unions of LBLs is identical to the family of unions of $\mathcal{Z}$-polyhedra.

Consider the image of the $\mathcal{Z}$-polyhedron represented as $\mathcal{Z} = \{Lz + l | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ by the affine lattice function represented as $(Kz' + k \rightarrow Rz' + r)$ where K has full column rank, m' say. By definition, the function provides a mapping from the iteration point $Kz' + k$ to the iteration point $Rz' + r$. The image $Rz' + r$ only exists for those values of z' for which the lattice point $Kz' + k$ lies in the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$. Thus, an image only exists for points in the intersection of the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ and the affine lattice $\{Kz' + k | z' \in \mathbb{Z}^{m'}\}$.

Consider the intersection of the affine lattices represented by $\mathcal{L} = \{Lz + l | z \in \mathbb{Z}^m\}$ and $\mathcal{L}' = \{Kz' + k | z' \in \mathbb{Z}^{m'}\}$. Let this be the affine lattice, say represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$ where L'' has full column rank. The affine lattice represented by $\mathcal{L}''$ may be empty, in which case the image is also empty. If $\mathcal{L}''$ represents a non-empty lattice, we have the following relationships.

$$L'' = LS, l'' = Ls + l$$
$$L'' = KS', l'' = Ks' + k$$

where S and S' are matrices and s and s' are vectors.

Taking the intersection of the $\mathcal{Z}$-polyhedron represented by $\mathcal{Z}$ and the affine lattice represented by $\mathcal{L}'' = \{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$, we get the following $\mathcal{Z}$-polyhedron.

$$\mathcal{Z} \cap \mathcal{L}'' = \{L''z'' + l'' | QSz'' + (q + Qs) \geq 0, z'' \in \mathbb{Z}^{m''}\} \tag{11}$$

An image by the function exists only for iterations in the $\mathcal{Z}$-polyhedron given above. Therefore, we may safely restrict the function to map iteration points in the lattice $\{L''z'' + l'' | z'' \in \mathbb{Z}^{m''}\}$. The restricted function is

$$(K(S'z'' + s') + k \rightarrow R(S'z'' + s') + r)$$

which is equivalent to

$$(L''z'' + l'' \rightarrow RS'z'' + (r + Rs'))$$

The image may be represented as

$$\{Tz'' + t | Q'z'' + q' \geq 0, z'' \in \mathbb{Z}^{m''}\} \tag{12}$$

where $T = RS'$, $t = (r + Rs')$, $Q' = QS$ and $q' = (q + Qs)$

The set in (12) is not necessarily a $\mathcal{Z}$-polyhedron since there is no guarantee on the rank of T. We will now provide an algorithm that transforms such a set into a union of $\mathcal{Z}$-polyhedra.

Let the context of the coordinate polyhedron $\mathcal{P} = \{z''|Q'z'' + q' \geq 0, z'' \in \mathbb{Z}^{m''}\}$ be given by $\ker(Q_0)$. If $\ker(A) \subseteq \ker(Q')$ where $A = \begin{pmatrix} T \\ Q_0 \end{pmatrix}$, the set in (12) is a $\mathcal{Z}$-polyhedron as a consequence of theorem 1.

When this condition fails, we pick any constant vector w in $\ker(A)\backslash\ker(Q')$. Let $\mathcal{P}'$ be the translation of $\mathcal{P}$ along v which equals either w or $-w$ such that

$$\mathcal{P} - \mathcal{P}' = \bigcup_{c_i \in \mathcal{C}} \mathcal{P} \cap (a_i^T z'' + \alpha_i < a_i^T v) \tag{13}$$

is a non-empty union, where $\mathcal{P}$ satisfies the set of constraints $\mathcal{C}$ and c_i is a constraint in $\mathcal{C}$ of the form $(a_i^T z'' + \alpha_i \geq 0)$. We are guaranteed that one of the translations (along w or $-w$) results in a non-empty union since $w \notin \ker(Q')$. The key insight into our proof is that the image of $\mathcal{P}$ by the affine lattice function equals the image of $\mathcal{P} - \mathcal{P}'$ by the affine lattice function. This is true since any element $z_1 \in \mathcal{P} \cap \mathcal{P}'$ is of the form $z_0 + \gamma v$ where $z_0 \in \mathcal{P} - \mathcal{P}'$ and γ is a constant. Since v lies in $\ker(T)$, its image satisfies the following property

$$Tz_1 + t = T(z_0 + \gamma v) + t = Tz_0 + t$$

For each non-empty element in the union of polyhedra in (13), create a union of polyhedra of the form

$$\mathcal{P}_{i,j} = \mathcal{P} \cap (a_i^T z'' + \alpha_i = \beta_j)$$

where $\beta_j \in \{0, \ldots, a_i^T v - 1)$. Now we claim that if the context of $\mathcal{P}_{i,j}$ is $\ker(Q_{0_{i,j}})$ then

$$\ker \begin{pmatrix} T \\ Q_{0_{i,j}} \end{pmatrix} \subset \ker \begin{pmatrix} T \\ Q_0 \end{pmatrix} \tag{14}$$

where the inclusion is strict. This is because

1. a_i^T is linearly independent of the rows of $\begin{pmatrix} T \\ Q_0 \end{pmatrix}$ since $\begin{pmatrix} T \\ Q_0 \end{pmatrix} v = 0$ and $a_0^T v \neq 0$.

2. a_i^T is a row of $Q_{0_{i,j}}$ and not a row of Q_0.

One iteration of this transformation returns an equivalent representation of the set in (12) that is a union of form

$$\bigcup_{i,j} \{Tz'' + t | Q'_{i,j} z'' + q'_{i,j} \geq 0, z'' \in \mathbb{Z}^{m''}\}$$

where $\mathcal{P}_{i,j} = \{z''|Q'_{i,j}z'' + q'_{i,j} \geq 0, z'' \in \mathbb{Z}^{m''}\}$

Thus, we are guaranteed that the algorithm will eventually terminate as a consequence of the strict inclusion presented in (14).

Finally, we wish to mention that this result does not violate the complexity results for deciding whether an LBL is a $\mathcal{Z}$-polyhedra since there can potentially be an exponential number of elements in our union.

6 Simplifying Reductions

The work presented in [8] shows the automatic and optimal decrease in the algorithmic complexity of reductions. It is one example of the extremely strong static analysis and program transformations offered by the polyhedral model. Here, we will show that the simplification of reductions can easily be extended to the $\mathcal{Z}$-polyhedral model, as a consequence of our representation and the constructive proofs of the closure properties.

For simplification, a reduction is required to be defined over an expression of the form $E.f$ where the domain of the expression is a single polyhedron, and projected by a standard affine function. By the closure of $\mathcal{Z}$-polyhedral domains under set difference, more specifically, by the algorithm presented as its constructive proof, we express any arbitrary $\mathcal{Z}$-polyhedral domain (of the expression) as a disjoint union of $\mathcal{Z}$-polyhedra. This is semantically equivalent to an expression that has, as subexpressions, reductions over expressions defined on elements ($\mathcal{Z}$-polyhedra) in the disjoint domain. Simplification of the original reduction is then simply the decrease of the asymptotic complexity these "smaller" reductions.

Now, let us consider one of these simpler reductions. With the techniques presented in the constructive proofs for closure under image and preimage, we may derive an equivalent reduction with the following properties.

1. The domain of the expression within the reduction is represented by $\mathcal{Z} = \{Lz+l | Qz+q \geq 0, z \in \mathbb{Z}^m\}$
2. The dependence function, f, is represented by $(Lz + l \rightarrow Rz + r)$
3. The projection function is represented as $(Lz + l \rightarrow Tz + t)$

The simpler reduction is equivalent to the reduction over an expression whose domain is $\{z | Qz + q \geq 0, z \in \mathbb{Z}^m\}$ and whose value at z is the value of the original expression at $Lz + l$. The associated dependences and projections are of the form $(z \rightarrow Rz + r)$ and $(z \rightarrow Tz + t)$ respectively. This reduction is in the form required for simplification.

In this example, we presented the generalization of an analysis developed for the polyhedral model to the $\mathcal{Z}$-polyhedral model. However, an important observation is that the generalization was performed through the transformation of the analysis in the $\mathcal{Z}$-polyhedral model to the original analysis in the polyhedral model. This shows that, in many cases, one may reuse techniques and tools developed for the polyhedral model.

7 Related Work

The first work that proposed the extension to a language based on unions of $\mathcal{Z}$-polyhedra was by Quinton et. al. [16]. However, as a consequence of their representation and interpretation, they did not have a unique canonic representation. Also they could not establish the equivalence between identical $\mathcal{Z}$-polyhedra nor did they provide the difference or two $\mathcal{Z}$-polyhedra. Other consequences included complex semantics for change of basis. In many ways, our paper is a logical completion of their efforts initiated a decade ago.

Ramanujam [17] describes algorithms to generate code, both sequential and parallel, after applying non-unimodular transformations to nested loop programs. His work is restricted to a single, perfectly nested loop nest, and the same transformation is applied to all the statements in the loop body. The code generation problem thus reduced to scanning the image, by a non-unimodular function, of a single polyhedron.

Rajopadhye and Lenders [12] propose a technique for designing multi-rate VLSI arrays, which are regular arrays of processing elements, but where different registers are clocked at different rates. This leads to very efficient hardware structures. The mathematical formalism is based on using systems of recurrence equations (i.e., equational programs) defined over $\mathcal{Z}$-polyhedral domains, which are viewed as the images of polyhedra by non-singular affine transformations. Although the focus of the paper is on synthesis methods, notably scheduling and localization, the authors discuss the "legality" of the proposed specification, in terms of checking whether a variable is actually defined at all points in the domain where it is declared. This requires determining whether the values of other variables specified on the right hand side of the equation are defined at precisely those points, which requires the closure properties we describe here. Rajopadhye and Lenders provide sufficient conditions, not a complete solution.

8 Conclusions and Future Work

It has been believed for more than a decade that the polyhedral model can be generalized to unions of $\mathcal{Z}$-polyhedra [16, 11]. However, till date, previously known theories and tools on the polyhedral model have not been generalized to unions of $\mathcal{Z}$-polyhedra.

We present a novel representation and interpretation of $\mathcal{Z}$-polyhedra that enables us to prove the various closure properties of the family of unions of $\mathcal{Z}$-polyhedra required to extend the polyhedral model. In addition, we prove closure in the $\mathcal{Z}$-polyhedral model under images by arbitrary affine functions which had been a major limitation of the polyhedral model. As a corollary, we prove that unions of LBLs, widely assumed to be a richer class, are equivalent to unions of $\mathcal{Z}$-polyhedra.

The language-theoretic aspect of the $\mathcal{Z}$-polyhedral model is also very interesting. Our equational language is purely functional, and through its incorporation into a general purpose functional language, one may make decades of research in the automatic parallelization available to modern functional languages.

Future work involves the extension of the various techniques in the polyhedral model. We intend to provide an implementation for manipulating unions of $\mathcal{Z}$-polyhedra based on our results. The canonic representation of *unions* of $\mathcal{Z}$-polyhedra is also an open problem.

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