Exact Solution to the Governing PDE of a Hot Water to Air Finned Tube Cross Flow Heat Exchanger

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Abstract

A new coil dynamic model is presented that utilizes the exact solution to the coil governing partial differential equation for a step change in water flow rate. This new model is the first step to developing a future model that can accurately predict the coil dynamics for several varying coil inlet conditions expected to occur under MIMO control. The new model is compared with previously published simplified PDE coil models and against actual measured coil dynamics. Several advantages of this new coil model are discussed.

1 Introduction

This report investigates a dynamic heat exchanger model derived from the first principles of thermodynamics. Previously published dynamic models for heat exchangers have been discrete time models (DTMs) or simplified first principle derivations. This section begins by analytically solving the governing partial differential equation for a cross flow, fluids unmixed, finned tube heat exchanger. It concludes by comparing the dynamic prediction to actual dynamic performance of the heat exchanger for a step change in water flow rate. The model presented is the first step in the development of a PDE model that can predict the coil dynamics under several simultaneous changing inlet conditions. This future model will be helpful for the simulation of more complex HVAC control schemes.

Dynamic models of a cross flow heat exchanger were first presented by Gartner and Harrison in 1965 [1]. Their model as well as other published models that followed primarily investigated the frequency response of heat exchangers [2][3][4]. Other models solved the PDE of one finned element of the cross flow heat exchanger and used the solution to these elements to determine the solution of the heat exchanger [5][6]. The computation time of these finite element solutions was fairly lengthy. Still other models considered a discrete time solution of the heat exchanger dynamics [7].

Tamm was the first to develop a dynamic multi-row counterflow coil model [8]. His model, like Gartner and Harrison was interpreted in the frequency domain. The terms needed in the solution grew exceedingly numerous as the number of coil passes increased. The finite element models of B.A. Reichert and associates [5] as well as S. Kabelac [6] investigate multi-pass heat exchangers as well. Both publications address the lengthy amount of time that counter flow arrangement models need to converge on a solution.

The master's thesis of McCutchan investigated the time solution of a cross flow, water to air heat exchanger [9]. His thesis extended the work of Gartner and Harrison by developing a first principles model of a finned serpentine cross flow heat exchanger. The mixed partial differential equation that resulted was considered too difficult to solve when McCutchan's research was published. Instead, McCutchan divided the dynamics of the coil into two separate actions and used superposition to determine model predictions.

2 PDE Model

In order to simulate complex HVAC control schemes such as MIMO (Multi Input Multi Output) controllers that utilize several changing heating coil inlet conditions at the same time, more complex coil dynamic models must be developed. The dynamic model presented here is the first step in developing a more complex model and is an extension of the model presented in the paper by Pearson, Leonard, and [10]. Their model is developed for a single pass, cross flow, hot water to air, finned tube heat exchanger but can also be extended to a multi pass heat exchanger such as the one used in this study. The partial differential equation model discussed but not solved in their paper was developed from first principle energy balances. This model looks at the coil dynamics for the case of a step change in hot water flow rate initially having no flow and no temperature gradient from the coil water to the air flowing across the coil.

Assumptions:

1. The densities and specific heats of the tube material, fin material, water, and air are considered to be constant and are evaluated at their mean value.

2. The heat capacitive effects of the water and metal contained in the U-tube bends are accounted for by distributing the U-bend metal and water throughout the finned portion of the coil.

3. Convective heat transfer coefficients on the air and water sides are independent of temperature, time, location and are evaluated at their mean temperature.

4. Conductive resistance through the tube wall is negligible.

5. Thermal resistance between the tube and fins is negligible.

6. Heat conduction in the water and tube in the axial direction is negligible.

7. Conduction through the fins from row to row is negligible.

8. Air temperature and velocity are constant throughout the entrance cross section to the heat exchanger.

9. The effective temperature difference between the metal and air for the heat transfer purposes is based upon the log mean temperature difference between the metal and air.

10. Fin effectiveness is constant.

Each run of the heat exchanger can now be modeled as a long finned tube heat exchanger as shown by Figure 1. The energy balance across an element

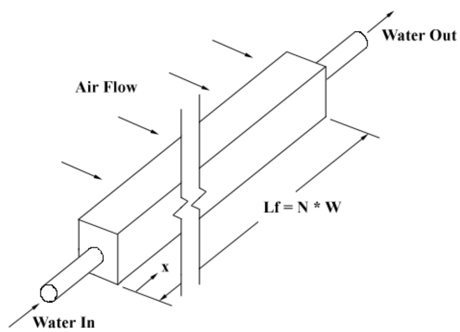

Figure 1: Straight Line Model of Heat Exchanger

of the heat exchanger shown in figure 2 leads to the set of three equations with three unknowns (T_{ao}, T_t, T_w) describing the transient heat transfer process.

$$\dot{m}_w c_{pw} \frac{\partial T_w}{\partial x} + \rho_w c_{pw} A'_{cw} \frac{\partial T_w}{\partial t} \frac{L_t}{L_f} + h_w A'_w (T_w - T_t) = 0 \tag{1}$$

$$\dot{m}'_a c_{pa} (T_{ao} - T_{ai}) - h_w A'_w (T_w - T_t) + c'_m \frac{\partial T_t}{\partial t} = 0 \tag{2}$$

$$\dot{m}'_a c_{pa} (T_{ao} - T_{ai}) = \eta_o h_a A'_a LMTD \tag{3}$$

Defining dimensionless B parameters as

$$B_1 \equiv \frac{\dot{m}_w c_{pw}}{\dot{m}'_a c_{pa} L_f} \qquad B_3 \equiv \frac{c'_m}{\dot{m}'_a c_{pa} FT}$$

$$B_2 \equiv \frac{h_w A'_w}{\dot{m}'_a c_{pa}} \qquad B_4 \equiv \frac{\eta_o h_a A'_a}{\dot{m}'_a c_{pa}}$$

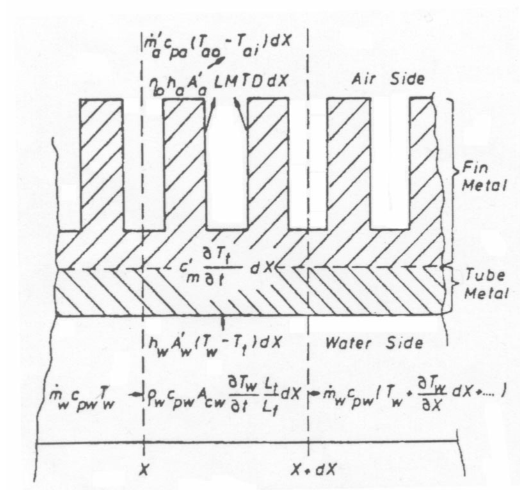

Figure 2: Thermal Energy Balance on an Element of the Straight Line Heat Exchanger Model

Lets equations 1, 2, and 3 to be expressed in dimensionless form as

$$B_1 \frac{\partial \theta_w}{\partial x^*} + B_2(\theta_w - \theta_t) + B_1 \frac{\partial \theta_w}{\partial t^*} = 0 \tag{4}$$

$$\theta_{xao} - B_2(\theta_w - \theta_t) + B_3 \frac{\partial \theta_t}{\partial t^*} = 0 \tag{5}$$

$$\theta_{xao} = B_4 LM\theta D \tag{6}$$

Where solving $LM\theta D$ for θ_t gives and additional relationship

$$\theta_{xao} = C_4 \theta_t \tag{7}$$

Combining equations 4 through 7 yields the mixed partial differential equation that describes the outlet air temperature.

$$\frac{B_1 B_3}{B_2 C_4} \frac{\partial}{\partial t^*} \left(\frac{\partial \theta_{xao}}{\partial t^*} + \frac{\partial \theta_{xao}}{\partial x^*} \right) + \left(\frac{B_3}{C_4} + \frac{B_1}{C} \right) \frac{\partial \theta_{xao}}{\partial t^*} + \frac{B_1}{C} \frac{\partial \theta_{xao}}{\partial x^*} + \theta_{xao} = 0 \tag{8}$$

Equation 8 is the same as equation 5 printed in the Pearson paper. In order to solve this PDE, the boundary and initial conditions must be specified.

In order to solve the boundary condition, the initial air outlet temperature distribution must be known at the point $x^* = 0$. The solution at this point can be calculated by solving the energy balance at the inlet of the coil for a steady state condition. Using the energy balance shown in figure 2 for steady state, the energy balance yields

$$\theta_{xao}(0,0) = C_i \tag{9}$$

The boundary condition at $x^* = 0$ is obtained by solving equation 4 with the help of equation 7

$$\theta_{xao}(0, t^*) = C_f + (C_i - C_f)\exp(-(\frac{C_{4f} + B_{2f}}{B_{3f}})t^*) \tag{10}$$

Solving for the initial air outlet temperature distribution to get the first initial condition, the time derivatives in equation 8 are set to zero then θ_{xao} is solved for yielding

$$\theta_{xao} = C_i \exp(-b_i x^*) \tag{11}$$

The second initial condition, $\frac{\partial \theta_{xao}(x^*,0)}{\partial t^*}$ is equal to 0 because the coil is at steady state at t=0

$$\frac{\partial \theta_{xao}(x^*, 0)}{\partial t^*} = 0 \tag{12}$$

The form of equation 8 can be written in a way that makes it easier to work with. Let $u = \theta_{xao}$ and the derivatives be set as subscripts of u. Also divide the constant on the second order derivatives through and set the new constants to simplified variables, let

$$a = (\frac{B_3}{C_4} + \frac{B_1}{C})\frac{B_2 C_4}{B_1 B_3},\ b = \frac{B_2 C_4}{C B_3},\ c = \frac{B_2 C_4}{B_1 B_3} \tag{13}$$

Equation 8 can now be written as

$$u_{tt} + u_{tx} + a u_t + b u_x + c u = 0 \tag{14}$$

With boundary value and initial conditions as

$$u(0, t^*) = C_f + (C_i - C_f)\exp(-(\frac{C_{4f} + B_{2f}}{B_{3f}})t^*) = f(t^*) \tag{15}$$

$$u(x^*, 0) = C_i \exp(-b_i x^*) = g(x^*) \tag{16}$$

$$u_t(x^*, 0) = 0 = h(x^*) \tag{17}$$

3 PDE Solution

The approach taken in this study is to separate the PDE given in equation 15 into a boundary value problem and an initial-boundary value problem then combining the solutions of these sub-problems by superposition to obtain the general solution. Because the dynamic case that this study investigates incorporates zero initial conditions, only the solution to the boundary value problem is implemented in the following chapters. The solution to the initial-boundary value problem is still presented as a matter of completion for future dynamic coil models.

For a non-zero boundary value condition and zero initial conditions, sub-problem 1 is defined as

$$\begin{aligned} u_{tt} + u_{tx} + au_t + bu_x + cu = 0 \\ u(0, t^*) = f(t^*) \\ u(x^*, 0) = 0 \\ u_t(x^*, 0) = 0 \end{aligned} \tag{18}$$

From here on down, the non-dimensional superscript * is dropped from all x and t variables.

To simplify sub-problem 1, we let

$$u = ve^{((2b-a)x-bt)} \tag{19}$$

Equation 19 becomes

$$v_{tt} + v_{tx} + Av = 0 \qquad x \geq 0, t \geq 0, A = c - ab + b^2 \tag{20}$$

Taking the LaPlace transform and solving with the boundary condition gives

$$v(x^*, s) = \tilde{F}(s)e^{-\frac{s^2+A}{s}x} \tag{21}$$

Using the LaPlace identities

$$e^{-xs}\bar{f}(s) \rightarrow \begin{cases} 0, & \text{for } t < x \\ f(t-x), & \text{for } t \geq 0 \end{cases} \tag{22}$$

$$L^{-1}[s\tilde{F}(s)] = \tilde{F}'(t) \tag{23}$$

$$L^{-1}[\frac{1}{s}e^{-\frac{Ax}{s}}] = J_o(2\sqrt{Axt}) \tag{24}$$

$$L^{-1}(\bar{f}(s)\bar{g}(s)) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{25}$$

gives

$$v(x,t) \rightarrow \begin{cases} 0, & \text{for } t < x \\ \int_0^{t-x} \tilde{F}'(t-x-\tau)J_o(2\sqrt{Ax\tau})d\tau, & \text{for } t \geq x \end{cases} \tag{26}$$

Using equation 26 and 19, the solution to sub-problem 1 becomes

$$u_1(x,t) = v(x,t)e^{((2b-a)x-bt)} \tag{27}$$

For a boundary value condition forced to 0 and non-zero initial conditions, sub-problem 2 is defined as

$$\begin{aligned} u_{tt} + u_{tx} + au_t + bu_x + cu = 0 \\ u(0, t) = 0 \\ u(x, 0) = g_o(x) \\ u_t(x, 0) = h_o(x) \end{aligned} \tag{28}$$

Where $g_o(x)$ and $h_o(x)$ are the odd extensions of the initial conditions $g(x)$ and $h(x)$.

Using the transformations $\hat{x} = 2x - t$ and $\hat{t} = t$, sub-problem 2 becomes

$$\begin{aligned} U_{\hat{t}\hat{t}} - U_{\hat{x}\hat{x}} + AU_{\hat{t}} + BU_{\hat{x}} + CU &= 0 \\ U(\hat{x}, 0) &= G(\hat{x}) \\ U_{\hat{t}}(\hat{x}, 0) &= H(\hat{x}) \end{aligned} \tag{29}$$

Where $A = a$, $B = 2b - a$, $C = c$.

The solution to equation 29 has already been presented by Guenther and Lee in their book, "Partial Differential Equations of Mathematical physics and Integral Equations", pp. 114-121 [11]. The solution is represented here with a few notes on how the solution needs to be implemented for a cross flow heat exchanger.

Letting,

$$\alpha = \hat{x} + \hat{t} = 2x \qquad \beta = \hat{x} - \hat{t} = 2(x - t)$$
$$\lambda = \frac{-A + B}{4} \qquad \mu = \frac{A + B}{4}$$

and

$$u(\frac{1}{2}(\alpha + \beta), \frac{1}{2}(\alpha - \beta)) = w(\alpha, \beta) \exp[\lambda\alpha + \mu\beta] \tag{30}$$

then

$$\begin{aligned} w_{\alpha\beta} &= -kw \\ w(\alpha, \alpha) &= G(\alpha) \exp[-\frac{|\alpha|\beta}{2}] \equiv \tilde{G}(\alpha) \\ w_\alpha(\alpha, \alpha) &= \frac{1}{2}[\tilde{G}'(\alpha) + J(\alpha)] \equiv \phi(\alpha) \\ w_\beta(\alpha, \alpha) &= \frac{1}{2}[\tilde{G}'(\alpha) - J(\alpha)] \equiv \psi(\alpha) \\ w_\alpha(\alpha, \alpha) - w_\beta(\alpha, \alpha) &= [\frac{A}{2}G(\alpha) + H(\alpha)] \exp[-\frac{|\alpha|\beta}{2}] \\ &\equiv J(\alpha) \end{aligned} \tag{31}$$

where

$$k = -[\frac{C}{4} + \lambda\mu] \tag{32}$$

Note: the term $\exp[-\frac{|\alpha|\beta}{2}]$ has to be written as an even extension in order to keep the initial conditions as bounded odd extensions. The Guenther book has this written as $\exp[-\frac{\alpha\beta}{2}]$ [11]. Now define

$$\Phi(\alpha, \beta) \equiv \frac{1}{2}[\tilde{G}(\alpha) + \tilde{G}(\beta)] + \frac{1}{2}\int_\beta^\alpha \phi(\xi)d\xi - \frac{1}{2}\int_\beta^\alpha \psi(\eta)d\eta \tag{33}$$

Then the solution to 32 is

$$w(\alpha,\beta) = \Phi(\alpha,\beta) + k\int_{\beta}^{\alpha}\int_{\eta}^{\alpha} I_o(2\sqrt{k(\alpha-\xi)(\eta-\beta)})\Phi(\xi,\eta)d\xi d\eta \qquad (34)$$

Applying equation 30, the solution to 29 is

$$u_2(x,t) = w(2x, 2(x-t))e^{[\lambda 2x + \mu 2(x-t)]} \qquad (35)$$

By superposition, equations 27 and 35 add to form the general solution to 15

$$u(x,t) = u_1(x,t) + u_2(x,t) \qquad (36)$$

4 PDE Model Results

Figure 3 shows several coil model predictions to a step in water velocity from 0% to 20% valve opening for coils initially having no water flow and a zero initial temperature gradient between the water and air. The comparison was done for an air flow rate of 0.5 m^3/s and inlet air temperature around 25 ^oC. In order to compare the PDE model to actual experimental results, the temperature along the coil length was averaged for each time step to produce a data set of coil outlet air temperature versus time only. Another adjustment needed in order to compare the models with the measured coil dynamics is to incorporate the air temperature sensor dynamics into the model predictions. Because the air temperature sensor time constant ranged from 33 seconds at a low air flow rate to 37 seconds at a high air flow rate, a time constant of 35 seconds was chosen to be used in the air temperature sensor filter. The model predictions were thus filtered through a first order transfer function that had a 35 second time constant.

There are three dynamic models that are compared with experimental data in figure 3. The first model is the PDE model and its development and use has been discussed in the previous sections. The P&L mixed beta model and the P&L final beta model are models developed in the Pearson, Leonard, and McCutchan paper and are known as simplifications of the general PDE solution [10]. The P&L mixed beta model uses both final and initial coil property values (ie. beta values) to predict coil dynamics while the P&L final beta model uses only final coil property values to predict coil dynamics. The main advantage of both of these simplified coil models is that they are easy to use.

For this valve position change, the PDE model predicted the dynamics almost perfectly with small differences noticed only near the beginning of the valve change at 30 seconds. The P&L models predict the dynamics within 2 ^oC for all time but the P&L mixed beta model predicts a response time of almost 100 seconds longer than the actual response time (here response time is the time it takes for the temperature to go from +0.5 ^oC of the initial air outlet temperature to -0.5 ^oC of the final air outlet temperature).

Along with the accuracy of the PDE model, the PDE model has significant advantages over other simplified coil models. First, the PDE model is an exact solution that is valid for all ranges of coil inlet conditions while simplified models are not valid beyond the range of their simplified assumptions. In other

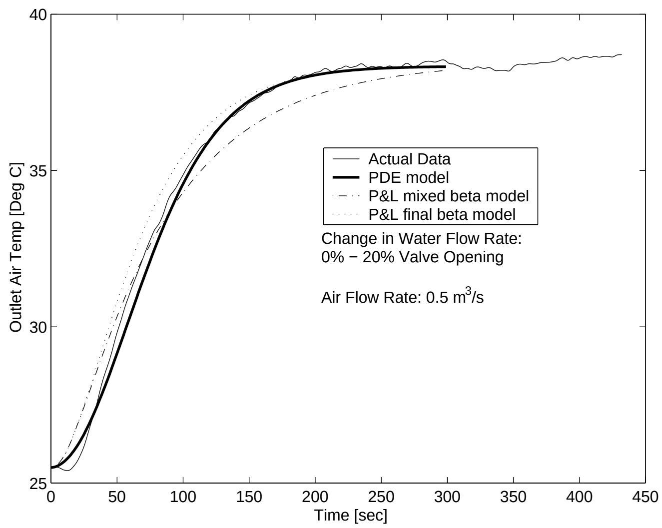

Figure 3: Dynamic Comparisons of Coil Model Predictions and Real Experimental Data for a Valve Change of 0% to 20% Open

words, the PDE model is not restricted by as many assumptions as simplified coil models are and it is valid for a wider range of inlet conditions than the simplified models are. Another advantage of the PDE model is that the solution is presented for time and distance along the coil while the models are only solved for time. This added spatial solution allows for more complex analysis between the coil model and the actual coil dynamics. Finally, the PDE model can be solved for various boundary conditions and initial conditions while the models are only valid for zero initial conditions and a step change in water flow rate boundary condition.

5 Conclusion

The dynamic heating coil PDE model presented in this report is the first step in the development of a dynamic coil model that can give coil predictions for simultaneously varying coil inlet conditions expected under action of a MIMO controller. The model presented here shows improvement over previous simplified PDE coil models as well as being applicable to a wider range of coil initial and boundary conditions. In general, the PDE model presented here is a significant first step for future exact PDE coil models.

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