NP-complete problems

A variety of NP-complete problems

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

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Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle $\Gamma$ that contains every node in $V$.

YES: vertices and faces of a dodecahedron.

NO: bipartite graph with odd number of nodes.

Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a directed graph $G = (V, E)$, does there exist a simple directed cycle $\Gamma$ that contains every node in $V$?

 Claim. $G$ has a Hamiltonian cycle iff $G'$ does.

Proof. Given a directed graph $G = (V, E)$, construct an undirected graph $G'$ with $3n$ nodes.

Suppose $G$ has a directed Hamiltonian cycle $\Gamma$.

Then $G'$ has an undirected Hamiltonian cycle (same order).

Suppose $G'$ has an undirected Hamiltonian cycle $\Gamma'$.

$\Gamma'$ must visit nodes in $G'$ using one of following two orders:

$\cdots, B, R, G, B, R, G, \cdots$

or

$\cdots, B, G, R, B, R, G, \cdots$

Blue nodes in $\Gamma'$ make up directed Hamiltonian cycle $\Gamma$ in $G$, or reverse of one.

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3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-SAT \(\leq_P\) DIR-HAM-CYCLE.

Proof. Given an instance \(\Phi\) of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff \(\Phi\) is satisfiable.

Construction. First, create a graph that has \(2^n\) Hamiltonian cycles which correspond in a natural way to \(2^n\) possible truth assignments.

For each clause, add a node and 6 edges.

Claim. \(\Phi\) is satisfiable iff \(G\) has a Hamiltonian cycle.

Proof. \(\Rightarrow\)
- Suppose 3-SAT instance has satisfiable assignment \(x\).
- Then, define Hamiltonian cycle in \(G\) as follows:
  - if \(x_i = 1\), traverse row \(i\) from left to right
  - if \(x_i = 0\), traverse row \(i\) from right to left
  - for each clause \(C_j\), there will be at least one row in which we are going in the "correct" direction to include node \(C_j\)

\(3-SAT\) Reduces to Directed Hamiltonian Cycle

Traveling Salesperson Problem

TSP. Given a set of \(n\) cities and a distance function \(d(u, v)\), is there a tour of length \(\leq D\)?
Traveling Salesperson Problem

TSP: Given a set of $n$ cities and a distance function $d(u, v)$, is there a tour of length $\leq D$?

Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Graph Coloring

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K-Coloring and Register Allocation

Generalization: k-coloring
Arises in when trying to allocate resources in the presence of constraints

Register allocation. Assign program variables to machine register so that no more than $k$ registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between $u$ and $v$ if there exists an operation where both $u$ and $v$ are “live” at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is $k$-colorable.

3-Coloring

3-COLOR: Given an undirected graph $G$ does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

K-Coloring

Value of $k$ affects the difficulty of the problem:
- A graph is 2-colorable iff it is bipartite
- 3-Coloring is NP-complete.
**3-Coloring**

Claim. $3\text{-SAT} \leq P 3\text{-COLOR}$.

Proof. Given $3\text{-SAT}$ instance $\Phi$, we construct an instance of $3\text{-COLOR}$ that is 3-colorable iff $\Phi$ is satisfiable.

Initial construct: 

Properties:
- T, F, B each receive a different color, and literals receive the colors T, F
- The nodes for $x_i$ and $\bar{x}_i$ each receive a different color (T, or F)

**Gadget that represents a clause:**

If the clause is not satisfied the gadget is not 3-colorable

$$C_i = x_1 \lor \bar{x}_2 \lor \bar{x}_3$$

This node can't be colored!

If the clause is satisfied the gadget is 3-colorable

$$C_i = x_1 \lor \bar{x}_2 \lor \bar{x}_3$$

If the clause is satisfied the gadget is 3-colorable

$$C_i = x_1 \lor \bar{x}_2 \lor \bar{x}_3$$

**3-Coloring**

Claim. Graph is 3-colorable iff $\Phi$ is satisfiable.

Proof. Suppose graph is 3-colorable.
- Consider assignment that sets all T literals to true.
- (i) ensures each literal is T or F.
- (ii) ensures a literal and its negation are opposites.
3-Coloring

Claim. Graph is 3-colorable iff $\Phi$ is satisfiable.

Proof. Suppose graph is 3-colorable.
- (i) ensures each literal is T or F.
- (ii) ensures a literal and its negation are opposites.
- (iii) ensures at least one literal in each clause is T.

Planar 3-Colorability

PLANAR-3-COLOR. Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

Def. A graph is planar if it can be embedded in the plane in such a way that no two edges cross.

Applications: VLSI circuit design, computer graphics.

Kuratowski’s Theorem. An undirected graph $G$ is non-planar iff it contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$.
Planar 3-Colorability and Graph 3-Colorability

**Claim.** PLANAR-3-COLOR \(\leq_{P}\) PLANAR-GRAPH-3-COLOR.

**Proof sketch.** Create a vertex for each region, and an edge between regions that share a nontrivial border.

Planar k-Colorability

**PLANAR-2-COLOR.** Solvable in polynomial time.

**PLANAR-3-COLOR.** \(\text{NP-complete.}\)

**PLANAR-4-COLOR.** Solvable in \(O(1)\) time.

**Theorem.** [Appel-Haken, 1976] Every planar map is 4-colorable.

- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

**False intuition.** If PLANAR-3-COLOR is hard, then so is PLANAR-4-COLOR.

Polynomial-Time Reductions

- 3-SAT
- INDEPENDENT SET
- DIR-HAM-CYCLE
- VERTEX COVER
- SET COVER
- 2D-HAM-CYCLE
- GRAPH 3-COLOR
- HAM-CYCLE
- PLANAR 3-COLOR
- SCHEDULING
- SUBSET-SUM
- PACKING AND COVERING
- SEQUENCING
- PARTITIONING
- NUMERICAL

Dick Karp (1972)

1985 Turing Award