Inference Rules
(Rosen, Section 1.5)

TOPICS

• Logic Proofs
  ◦ via Truth Tables
  ◦ via Inference Rules

Propositional Logic Proofs

• An argument is a sequence of propositions:
  ◦ Premises (Axioms) are the first n propositions
  ◦ Conclusion is the final proposition.
• An argument is valid if \((p_1 \land p_2 \land \ldots \land p_n) \rightarrow q\) is a tautology, given that \(p_j\) are the premises (axioms) and \(q\) is the conclusion.
Proof Method #1: Truth Table

- If the conclusion is true in the truth table whenever the premises are true, it is proved
  - Warning: when the premises are false, the conclusion may be true or false
- Problem: given $n$ propositions, the truth table has $2^n$ rows
  - Proof by truth table quickly becomes infeasible

Example Proof by Truth Table

\[ s = ((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r) \]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>\neg p</th>
<th>p \lor q</th>
<th>\neg p \lor r</th>
<th>q \lor r</th>
<th>(p \lor q) \land (\neg p \lor r)</th>
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Proof Method #2: Rules of Inference

- A rule of inference is a pre-proved relation: any time the left hand side (LHS) is true, the right hand side (RHS) is also true.

- Therefore, if we can match a premise to the LHS (by substituting propositions), we can assert the (substituted) RHS

Inference properties

- Inference rules are truth preserving
  - If the LHS is true, so is the RHS

- Applied to true statements
  - Axioms or statements proved from axioms

- Inference is syntactic
  - Substitute propositions
    - if $p$ replaces $q$ once, it replaces $q$ everywhere
    - If $p$ replaces $q$, it only replaces $q$

  - Apply rule
**Example Rule of Inference**

**Modus Ponens**

\[ (p \land (p \rightarrow q)) \rightarrow q \]

\[ p \rightarrow q \]

\[ \therefore q \]

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
<th>( p \land (p \rightarrow q) )</th>
<th>( (p \land (p \rightarrow q)) \rightarrow q )</th>
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<tbody>
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**Rules of Inference**

<table>
<thead>
<tr>
<th>Rules of Inference</th>
<th>Modus Tollens</th>
<th>Hypothetical Syllogism</th>
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<tbody>
<tr>
<td><strong>Modus Ponens</strong></td>
<td><strong>Modus Tollens</strong></td>
<td><strong>Hypothetical Syllogism</strong></td>
</tr>
<tr>
<td>( p )</td>
<td>( \neg q )</td>
<td>( p \rightarrow q )</td>
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<td>( p \rightarrow q )</td>
<td>( p \rightarrow q )</td>
<td>( q \rightarrow r )</td>
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<td>( q )</td>
<td>( \neg p )</td>
<td>( p \rightarrow r )</td>
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<tr>
<td><strong>Addition</strong></td>
<td><strong>Resolution</strong></td>
<td><strong>Disjunctive Syllogism</strong></td>
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<tr>
<td>( p )</td>
<td>( p \lor q )</td>
<td>( p \lor q )</td>
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<td>( p \lor q )</td>
<td>( \neg p \lor r )</td>
<td>( \neg p )</td>
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<tr>
<td></td>
<td>( q \lor r )</td>
<td>( q )</td>
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<tr>
<td><strong>Simplification</strong></td>
<td><strong>Conjunction</strong></td>
<td></td>
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<tr>
<td>( p \land q )</td>
<td>( p )</td>
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<tr>
<td>( p )</td>
<td>( q )</td>
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<tr>
<td>( p \land q )</td>
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</tbody>
</table>
Logical Equivalences

**Idempotent Laws**
- \( p \lor p \equiv p \)
- \( p \land p \equiv p \)

**DeMorgan's Laws**
- \( \neg(p \land q) \equiv \neg p \lor \neg q \)
- \( \neg(p \lor q) \equiv \neg p \land \neg q \)

**Distributive Laws**
- \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)
- \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)

**Double Negation**
- \( \neg(\neg p) \equiv p \)

**Absorption Laws**
- \( p \lor (p \land q) \equiv p \)
- \( p \land (p \lor q) \equiv p \)

**Commutative Laws**
- \( p \lor q \equiv q \lor p \)
- \( p \land q \equiv q \land p \)

**Implication Laws**
- \( p \rightarrow q \equiv \neg p \lor q \)
- \( p \rightarrow q \equiv \neg q \rightarrow \neg p \)

**Associative Laws**
- \( (p \lor q) \lor r \equiv p \lor (q \lor r) \)
- \( (p \land q) \land r \equiv p \land (q \land r) \)

**Biconditional Laws**
- \( p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) \)
- \( p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p \)

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**Modus Ponens**

- If \( p \), and \( p \) implies \( q \), then \( q \)

Example:
- \( p = \) it is sunny, \( q = \) it is hot
- \( p \rightarrow q \), it is hot whenever it is sunny

“Given the above, if it is sunny, it must be hot”.
Modus Tollens

- If not q and p implies q, then not p

Example:

\[ p = \text{it is sunny}, \quad q = \text{it is hot} \]

\[ p \rightarrow q, \quad \text{it is hot whenever it is sunny} \]

“Given the above, if it is not hot, it cannot be sunny.”

Hypothetical Syllogism

- If p implies q, and q implies r, then p implies r

Example:

\[ p = \text{it is sunny}, \quad q = \text{it is hot}, \quad r = \text{it is dry} \]

\[ p \rightarrow q, \quad \text{it is hot when it is sunny} \]

\[ q \rightarrow r, \quad \text{it is dry when it is hot} \]

“Given the above, it must be dry when it is sunny”
Disjunctive Syllogism
- If \( p \) or \( q \), and not \( p \), then \( q \)

Example:
\( p = \text{it is sunny}, \ q = \text{it is hot} \)
\( p \lor q, \text{it is hot or sunny} \)
“Given the above, if it not sunny, but it is hot or sunny, then it is hot”

Resolution
- If \( p \) or \( q \), and not \( p \) or \( r \), then \( q \) or \( r \)

Example:
\( p = \text{it is sunny}, \ q = \text{it is hot}, \ r = \text{it is dry} \)
\( p \lor q, \text{it is sunny or hot} \)
\( \neg p \lor r, \text{it is not hot or dry} \)
“Given the above, if it is sunny or hot, but not sunny or dry, it must be hot or dry”

Not obvious!
Addition

- If p then p or q

Example:

\( p = \text{it is sunny}, \ q = \text{it is hot} \)

\( p \lor q, \text{it is hot or sunny} \)

“Given the above, if it is sunny, it must be hot or sunny”

Of course!

Simplification

- If p and q, then p

Example:

\( p = \text{it is sunny}, \ q = \text{it is hot} \)

\( p \land q, \text{it is hot and sunny} \)

“Given the above, if it is hot and sunny, it must be hot”

Of course!
Conjunction

- If p and q, then p and q

Example:
- p = it is sunny, q = it is hot
- p ∧ q, it is hot and sunny
- “Given the above, if it is sunny and it is hot, it must be hot and sunny”
- Of course!

A Simple Proof

Given X, X → Y, Y → Z, ¬ Z ∨ W, prove W

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1.</td>
<td>x → y</td>
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<td>2.</td>
<td>y → z</td>
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<td>3.</td>
<td>x → z</td>
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<td>4.</td>
<td>x</td>
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<td>5.</td>
<td>z</td>
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<td>6.</td>
<td>¬ z ∨ W</td>
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<tr>
<td>7.</td>
<td>W</td>
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</tbody>
</table>
A Simple Proof

“In order to sign up for CS161, I must complete CS160 and either M155 or M160. I have not completed M155 but I have completed CS161. Prove that I have completed M160.”

STEP 1) Assign propositions to each statement.
- A : CS161
- B : CS160
- C : M155
- D : M160

Setup the proof

STEP 2) Extract axioms and conclusion.
- Axioms:
  - $A \rightarrow B \land (C \lor D)$
  - $A$
  - $\neg C$
- Conclusion:
  - $D$
Now do the Proof

STEP 3) Use inference rules to prove conclusion.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Premise</td>
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<tr>
<td>2.</td>
<td>Premise</td>
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<td>3.</td>
<td>Modus Ponens (1, 2)</td>
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<td>4.</td>
<td>Simplification</td>
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<td>5.</td>
<td>Premise</td>
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<td>6.</td>
<td>Disjunctive Syllogism (4, 5)</td>
</tr>
</tbody>
</table>

Another Example

Given:  
\[ p \rightarrow q \]  
\[ \neg p \rightarrow r \]  
\[ r \rightarrow s \]

Conclude:  
\[ \neg q \rightarrow s \]
### Proof of Another Example

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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</thead>
<tbody>
<tr>
<td>1.</td>
<td>( p \rightarrow q ) Premise</td>
</tr>
<tr>
<td>2.</td>
<td>( \neg q \rightarrow \neg p ) Implication law (1)</td>
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<tr>
<td>3.</td>
<td>( \neg p \rightarrow r ) Premise</td>
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<tr>
<td>4.</td>
<td>( \neg q \rightarrow r ) Hypothetical syllogism (2, 3)</td>
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<tr>
<td>5.</td>
<td>( r \rightarrow s ) Premise</td>
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<tr>
<td>6.</td>
<td>( \neg q \rightarrow s ) Hypothetical syllogism (4, 5)</td>
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### Proof using Rules of Inference and Logical Equivalences

Prove: \( \neg(p \lor (\neg p \land q)) \equiv (\neg p \land \neg q) \)

\[
\begin{align*}
\neg (p \lor (\neg p \land q)) & \equiv \neg p \land \neg (\neg p \lor q) & & \text{By 2nd DeMorgan's} \\
& \equiv \neg p \land (\neg (\neg p) \lor \neg q) & & \text{By 1st DeMorgan's} \\
& \equiv \neg p \land (p \lor \neg q) & & \text{By double negation} \\
& \equiv (\neg p \land p) \lor (\neg p \land \neg q) & & \text{By double distributive} \\
& \equiv F \lor (\neg p \land \neg q) & & \text{By definition of \( \land \)} \\
& \equiv (\neg p \land \neg q) \lor F & & \text{By commutative law} \\
& \equiv (\neg p \land \neg q) & & \text{By definition of \( \lor \)}
\end{align*}
\]
Example of a Fallacy

\[
q \\
\therefore p
\]

\[
(q \land (p \rightarrow q)) \rightarrow p \\
p \rightarrow q
\]

<table>
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<tr>
<th>p</th>
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<th>p → q</th>
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<th>(q ∧ (p → q)) → p</th>
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This is not a tautology, therefore the argument is not valid.

Example of a Fallacy

- If q, and p implies q, then p

Example:

p = it is sunny, q = it is hot

p → q, if it is sunny, then it is hot

“Given the above, just because it is hot, does NOT necessarily mean it is sunny.