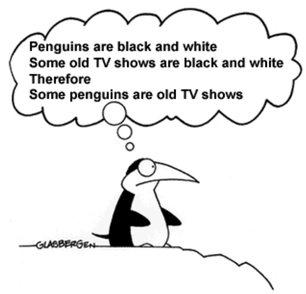
Propositional Logic, Truth Tables, and Predicate Logic
(Rosen, Sections 1.1, 1.2, 1.3)

TOPICS

- Propositional Logic
- Logical Operations
- Equivalences
- Predicate Logic

Logic?



What is logic?

Logic is a truth-preserving system of inference

Truth-preserving: If the initial statements are true, the inferred statements will be true

System: a set of mechanistic transformations, based on syntax alone

Inference: the process of deriving (inferring) new statements from old statements

Propositional Logic

- A *proposition* is a statement that is either true or false
- Examples:
  - This class is CS160 (true)
  - Today is Sunday (false)
  - It is currently raining in Singapore (???)
- Every proposition is true or false, but its *truth value* (true or false) may be unknown

Propositional Logic (II)

- A propositional statement is one of:
  - A simple proposition
    - denoted by a capital letter, e.g. ‘A’.
  - A negation of a propositional statement
    - e.g. ¬A : “not A”
  - Two propositional statements joined by a *connective*
    - e.g. A ∧ B : “A and B”
    - e.g. A ∨ B : “A or B”
  - If a connective joins complex statements, parenthesis are added
    - e.g. A ∧ (B∨C)

Truth Tables

- The truth value of a compound propositional statement is determined by its truth table
- Truth tables define the truth value of a connective for every possible truth value of its terms

Logical negation

- Negation of proposition A is ¬A
  - A: It is snowing.
  - ¬A: It is not snowing
  - A: Newton knew Einstein.
  - ¬A: Newton did not know Einstein.
  - A: I am not registered for CS195.
  - ¬A: I am registered for CS195.

Negation Truth Table

A	¬*A*
0	1
1	0

Logical and (*conjunction*)

- Conjunction of A and B is A ∧ B
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - A ∧ B: CS160 teaches logic and Java.
- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi: A ∧ ¬B

Truth Table for Conjunction

A	*B*	*A*∧*B*
0	0	0
0	1	0
1	0	0
1	1	1

Logical or (*disjunction*)

- Disjunction of A and B is A ∨ B
  - A: Today is Friday.
  - B: It is snowing.
  - A ∨ B: Today is Friday or it is snowing.
- This statement is true if any of the following hold:
  - Today is Friday
  - It is snowing
  - Both
- Otherwise it is false

Truth Table for Disjunction

A	*B*	*A* ∨*B*
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Or

- The "or" connective ∨ is inclusive: it is true if either *or both* arguments are true
- There is also an exclusive or ⊕

A	*B*	*A*⊕*B*
0	0	0
0	1	1
1	0	1
1	1	0

Confusion over Inclusive OR and Exclusive OR

- Restaurants typically let you pick one (either soup or salad, not both) when they say "The entrée comes with a soup or salad".
  - Use exclusive OR to write as a logic proposition
- Give two interpretations of the sentence using inclusive OR and exclusive OR:
  - Students who have taken calculus or intro to programming can take this class

Conditional & Biconditional Implication

- The conditional implication connective is →
- The biconditional implication connective is ↔
- These, too, are defined by truth tables

A	*B*	*A*→*B*
0	0	1
0	1	1
1	0	0
1	1	1

A	*B*	*A*↔*B*
0	0	1
0	1	0
1	0	0
1	1	1

Conditional implication

- A: A programming homework is due.
- B: It is Tuesday.
- A → B:
  - If a programming homework is due, then it must be Tuesday.
  - A programming homework is due only if it is Tuesday.
- Is this the same?
  - If it is Tuesday, then a programming homework is due.

Bi-conditional

- A: You can drive a car.
- B: You have a driver's license.
- A ↔ B
  - You can drive a car if and only if you have a driver's license (and vice versa).
- What if we said "if"?
- What if we said "only if"?

Compound Truth Tables

- Truth tables can also be used to determine the truth values of compound statements, such as $(A \lor B) \land (\neg A)$ (fill this as an exercise)

A	B	$\neg A$	$A \lor B$	$(A \lor B) \land (\neg A)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

Tautology and Contradiction

- A *tautology* is a compound proposition that is always true.
- A *contradiction* is a compound proposition that is always false.
- A *contingency* is neither a tautology nor a contradiction.
- A compound proposition is *satisfiable* if there is at least one assignment of truth values to the variables that makes the statement true.

Examples

A	$\neg A$	$A \lor \neg A$	$A \land \neg A$
0	1	1	0
1	0	1	0

Result is always true, no matter what A is

Therefore, it is a ***tautology***

Result is always false, no matter what A is

Therefore, it is a ***contradiction***

Logical Equivalence

- Two compound propositions, p and q, are logically equivalent if p ↔ q is a tautology.
- Notation: p ≡ q
- De Morgan's Laws:
  - ¬ (p ∧ q) ≡ ¬ p ∨ ¬ q
  - ¬ (p ∨ q) ≡ ¬ p ∧ ¬ q
- How so? Let's build a truth table!

Prove ¬(p ∧ q) ≡ ¬p ∨ ¬q

p	*q*	¬p	¬q	(p ∧ q)	¬(p ∧ q)	¬p ∨ ¬q
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

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Show ¬(p ∨ q) ≡ ¬p ∧ ¬q

p	*q*	¬p	¬q	(p ∨ q)	¬(p ∨q)	¬p ∧ ¬q
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

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Other Equivalences

- Show p → q ≡ ¬p ∨ q
- Show Distributive Law:
  - p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)

Show p → q ≡ ¬p ∨ q

p	*q*	¬p	p → q	¬p ∨ q
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Show p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)

p	*q*	*r*	q ∧ r	p ∨ q	p ∨ r	p ∨ (q ∧ r)	(p ∨ q) ∧ (p ∨ r)
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

More Equivalences

Equivalence	Name
p ∧ T ≡ p p ∨ F ≡ p	Identity
p ∧ q ≡ q ∧ p p ∨ q ≡ q ∨ p	Commutative
p ∨ (p ∧ q) ≡ p p ∧ (p ∨ q) ≡ p	Absorption

See Rosen for more.

Equivalences with Conditionals and Biconditionals

- Conditionals
  - p → q ≡ ¬p ∨ q
  - p → q ≡ ¬q → ¬p
  - ¬(p → q) ≡ p ∧ ¬q
- Biconditionals
  - p ↔ q ≡ (p → q) ∧ (q → p)
  - p ↔ q ≡ ¬p ↔ ¬q
  - ¬(p ↔ q) ≡ p ↔ ¬q

Prove Biconditional Equivalence

p	q	¬q	p ↔ q	¬(p ↔ q)	p ↔ ¬q
0	0	1	1	0	0
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0

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Converse, Contrapositive, Inverse

- The *converse* of an implication *p* → *q* reverses the propositions*:* ***q* → *p***
- The *inverse* of an implication *p* → *q* inverts both propositions: **¬*p* → ¬*q***
- The *contrapositive* of an implication *p* → *q* reverses and inverts: **¬*q* → ¬*p***

The converse and inverse are not logically equivalent to the original implication, but the contrapositive is, and may be easier to prove.

Predicate Logic

- Some statements cannot be expressed in propositional logic, such as:
  - All men are mortal.
  - Some trees have needles.
  - X > 3.
- Predicate logic can express these statements and make inferences on them.

Statements in Predicate Logic

P(x,y)

- Two parts:
  - A predicate P describes a relation or property.
  - Variables (x,y) can take arbitrary values from some domain.
- Still have two truth values for statements (T and F)
- When we assign values to x and y, then P has a truth value.

Example

- Let Q(x,y) denote "x=y+3".
  - What are truth values of:
    - Q(1,2) false
    - Q(3,0) true
- Let R(x,y) denote x beats y in Rock/Paper/Scissors with 2 players with following rules:
  - Rock smashes scissors, Scissors cuts paper, Paper covers rock.
  - What are the truth values of:
    - R(rock, paper) false
    - R(scissors, paper) true

Quantifiers

- Quantification expresses the extent to which a predicate is true over a set of elements.
- Two forms:
  - Universal ∀
  - Existential ∃

Universal Quantifier

- P(x) is true for all values in the domain
  ∀x∈D, P(x)
- For every x in D, P(x) is true.
- An element x for which P(x) is false is called a *counterexample*.
- Given P(x) as "x+1>x" and the domain of R, what is the truth value of:

  ∀x P(x) true

Example

- Let P(x) be that x>0 and x is in domain of R.
- Give a counterexample for:
  ∀x P(x)

x = -5

Existential Quantifier

- P(x) is true for <u>at least one value</u> in the domain.

  $\exists x \in D, P(x)$
- For some x in D, P(x) is true.
- Let the domain of x be "animals", M(x) be "x is a mammal" and E(x) be "x lays eggs", what is the truth value of:

  $\exists x\ (M(x) \land E(x))$

true
Platypuses
echidnas

English to Logic

- Some person in this class has visited the Grand Canyon.
- Domain of x is the set of all persons
- C(x): x is a person in this class
- V(x): x has visited the Grand Canyon
- $\exists x(C(x) \land V(x))$

English to Logic

- For every one there is someone to love.
- Domain of x and y is the set of all persons
- L(x, y): x loves y
- $\forall x \exists y\ L(x,y)$
- Is it necessary to explicitly include that x and y must be different people (i.e. $x \neq y$)?
  - Just because x and y are different variable names doesn't mean that they can't take the same values

English to Logic

- No one in this class is wearing shorts and a ski parka.
- Domain of x is <u>persons in this class</u>
  - S(x): x is wearing shorts
  - P(x): x is wearing a ski parka
  - $\neg\exists x(S(x) \land P(x))$
- Domain of x is all persons
  - C(x): x belongs to the class
  - $\neg\exists x(C(x) \land S(x) \land P(x))$

Evaluating Expressions: Precedence and Variable Bindings

- Precedence:
  - Quantifiers and negation are evaluated before operators
  - Otherwise left to right
- Bound:
  - Variables can be given specific values or
  - Can be constrained by quantifiers

Predicate Logic Equivalences

Statements are *logically equivalent* iff they have the same truth value under all possible bindings.

For example:

$$\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$$

In English: "Given the domain of students in CS160, all students have passed M124 course (P) and are registered at CSU (Q); hence, all students have passed M124 and all students are registered at CSU.

Other Equivalences

- Someone likes skiing (P) or likes swimming (Q); hence, there exists someone who likes skiing or there exists someone who likes skiing.

$$\exists x\big(P(x) \vee Q(x)\big) \equiv \exists x P(x) \vee \exists x Q(x)$$

- Not everyone likes to go to the dentist; hence there is someone who does not like to go to the dentist.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

- There does not exist someone who likes to go to the dentist; hence everyone does not like to go to the dentist.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$