Proof Techniques
(Rosen, Sections 1.5, 1.6, 1.7)

TOPICS

- Direct Proofs
- Proof by Contrapositive
- Proof by Contradiction
- Proof by Cases

Proof Terminology

Theorem: statement that can be shown to be true

Proof: a valid argument that establishes the truth of a theorem

Axioms: statements we assume to be true

Lemma: a less important theorem that is helpful in the proof of other results

Corollary: theorem that can be established directly from a theorem that has been proved

Conjecture: statement that is being *proposed* to be a true statement

Learning objectives

- Direct proofs
- Proof by contrapositive
- Proof by contradiction
- Proof by cases

Technique #1: Direct Proof

- Direct Proof:
  - First step is to clearly state the premise
  - Subsequent steps use rules of inference or other premises
  - Last step proves the conclusion

Direct Proof Example

- Prove "If n is an odd integer, then n^2 is odd."
  - If n is odd, then $n = 2k+1$ for some integer k.
  - $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
  - Therefore, $n^2 = 2(2k^2 + 2k) + 1$, which is odd.

2 * any # → even

Add 1 to any even # → odd #

More formal version…

	Step	Reason
1.	n is odd	Premise
2.	$\exists k \in Z \;\; n = 2k+1$	Def of odd integer in (1)
3.	$n^2 = (2k+1)^2$	Squaring (2)
4.	$= 4k^2 + 4k + 1$	Algebra on (3)
5.	$= 2(2k^2 + 2k) + 1$	Algebra on (4)
6.	$\therefore$ n^2 is odd	Def odd int, from (5)

Class Exercise

- Prove: If n is an even integer, then n^2 is even.
  - If n is even, then $n = 2k$ for some integer k.
  - $n^2 = (2k)^2 = 4k^2$
  - Therefore, $n = 2(2k^2)$, which is even.

Can you do the formal version?

	Step	Reason
1.	n is even	Premise
2.	$\exists k \in Z \;\; n = 2k$	Def of even integer in (1)
3.	$n^2 = (2k)^2$	Squaring (2)
4.	$= 4k^2$	Algebra on (3)
5.	$= 2(2k^2)$	Algebra on (4)
6.	$\therefore$ n^2 is even	Def even int, from (5)

Technique #2: Proof by Contrapositive

- A direct proof, but starting with the contrapositive equivalence:

  $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- If you are asked to prove $p \rightarrow q$, …
- …, you instead prove $\neg q \rightarrow \neg p$!
- Why? Sometimes, it may be easier to directly prove $\neg q \rightarrow \neg p$ than $p \rightarrow q$

Proof by Contrapositive Example

- Prove: If n^2 is an even integer, then n is even.

  (n^2 even) $\rightarrow$ (n even)
- By the contrapositive: This is the same as showing that

  $\neg$(n even) $\rightarrow$ $\neg$(n^2 even)
- If ***n*** is odd, then n^2 is odd. (proved on slides 4 and 5)
- Since we have proved the contrapositive:

  $\neg$(n even) $\rightarrow$ $\neg$(n^2 even)
- We have also proved the original hypothesis:

  (n^2 even) $\rightarrow$ (n even)

Technique #3: Proof by Contradiction

Prove: If *p* then *q*.

Proof strategy:

- Assume *p* and the negation of *q*.
- In other words, assume that $p \wedge \neg q$ is true.
- Then arrive at a contradiction $p \wedge \neg p$ (or something that contradicts a known fact).
- Since this cannot happen, our assumption must be wrong, thus, $\neg q$ is false. *q* is true.

Proof by Contradiction Example

Prove: If *(3n+2) is odd,* then *n is odd*.

Proof:

- Given: (3n+2) is odd.
- Assume that n is not odd, that is n is even.
- If n is even, there is some integer k such that n=2k.
- (3n+2) = (3(2k)+2)=6k+2 = 2(3k+1), which is 2 times a number.
- Thus 3n+2 turned out to be even, but we know it's odd.
- This is a contradiction. Our assumption was wrong.
- Thus, n must be odd.

Proof by Contradiction Example

Prove that the $\sqrt{2}$ is irrational.

- Assume that $\sqrt{2}$ is not irrational, i.e. $\sqrt{2}$ is rational.
- Hence, $\sqrt{2} = \frac{a}{b}$ and a and b have no common factors.

(Rational definition, fraction must be in lowest terms.)

- So $a^2 = 2b^2$ which means a is even, hence $a = 2c$
- Therefore, $b^2 = 2c^2$ then b must be even
- So a and b have at least the common factor 2
- Contradiction, so $\sqrt{2}$ is irrational after all!

Technique #4: Proof by Cases

- Given a problem of the form:
  - $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$
  - where $p_1, p_2, \ldots p_n$ are the cases
- This is equivalent to the following:
  - $[(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$
- So prove all the clauses are true.

Proof by Cases Example

- Prove: If n is an integer, then $n^2 \geq n$
  - $(n = 0 \vee n \geq 1 \vee n \leq -1) \rightarrow n^2 \geq n$
- Show for all the three cases, i.e.,
  - $(n = 0 \rightarrow n^2 \geq n) \wedge (n \geq 1 \rightarrow n^2 \geq n)$
    $\wedge\ (n \leq -1 \rightarrow n^2 \geq n)$

Proof by Cases (cont'd)

- Case 1: Show that $n = 0 \rightarrow n^2 \geq n$
  - When n=0, $n^2= 0$.
  - $0 \geq 0$ ☺
- Case 2: Show that $n \geq 1 \rightarrow n^2 \geq n$
  - Multiply both sides of the inequality by n
  - We get $n^2 \geq n$

Proof by Cases (cont'd)

- Case 3: Show that $n \leq -1 \rightarrow n^2 \geq n$
  - Given $n \leq -1$,
  - We know that n^2 cannot be negative, i.e., $n^2 > 0$
  - We know that $0 > -1$
  - Thus, $n^2 > -1$. We also know that $-1 \geq n$ (given)
  - Therefore, $n^2 \geq n$

Proof by Cases
Can you finish this?

Theorem: Given two real numbers *x* and *y*,

$$abs(x*y)=abs(x)*abs(y)$$

Exhaustively determine the premises

Case p1: x>=0, y>=0

Proof: x*y>=0 so abs(x*y)=x*y and abs(x)=x and abs(y)=y so abs(x)*abs(y)=x*y

Case p2: x<0, y>=0

Case p3: x>=0, y<0

Case p4: x<0, y<0