



Sets and Functions (Rosen, Sections 2.1,2.2, 2.3)

TOPICS

- Discrete math
- Set Definition
- Set Operations
- Tuples



Why Study Discrete Math?

- Digital computers are based on discrete units of data (bits).
- Therefore, both a computer's
 - structure (circuits) and
 - operations (execution of algorithms)can be described by discrete math
- A generally useful tool for rational thought! Prove your arguments.



What is 'discrete'?

- Consisting of distinct or unconnected elements, not continuous (calculus)
- Helps us in Computer Science:
 - What is the probability of winning the lottery?
 - How many valid Internet address are there?
 - How can we identify spam e-mail messages?
 - How many ways are there to choose a valid password on our computer system?
 - How many steps are need to sort a list using a given method?
 - How can we prove our algorithm is more efficient than another?



Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, etc....
- *i.e.*, the whole field!



What is a set?

- An *unordered collection of objects*
 - $\{1, 2, 3\} = \{3, 2, 1\}$ since sets are unordered.
 - $\{a, b, c\} = \{b, c, a\} = \{c, b, a\} = \{c, a, b\} = \{a, c, b\}$
 - $\{2\}$
 - $\{\text{on, off}\}$
 - $\{\}$



What is a set?

- Objects are called *elements* or *members* of the set
- Notation \in
 - $a \in B$ means "a is an element of set B."
 - Lower case letters for elements in the set
 - Upper case letters for sets
 - If $A = \{1, 2, 3, 4, 5\}$ and $x \in A$, what are the possible values of x?



What is a set?

- **Infinite Sets** (*without end, unending*)
 - $N = \{0, 1, 2, 3, \dots\}$ is the Set of natural numbers
 - $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the Set of integers
 - $Z^+ = \{1, 2, 3, \dots\}$ is the Set of positive integers
- **Finite Sets** (*limited number of elements*)
 - $V = \{a, e, i, o, u\}$ is the Set of vowels
 - $O = \{1, 3, 5, 7, 9\}$ is the Set of odd #'s < 10
 - $F = \{a, 2, \text{Fred, New Jersey}\}$
 - Boolean data type used frequently in programming
 - $B = \{0, 1\}$
 - $B = \{\text{false, true}\}$
 - Seasons = $\{\text{spring, summer, fall, winter}\}$
 - ClassLevel = $\{\text{Freshman, Sophomore, Junior, Senior}\}$



What is a set?

- **Infinite vs. finite**
 - If finite, then the number of elements is called the *cardinality*, denoted $|S|$
 - $V = \{a, e, i, o, u\}$ $|V| = 5$
 - $F = \{1, 2, 3\}$ $|F| = 3$
 - $B = \{0, 1\}$ $|B| = 2$
 - $S = \{\text{spring, summer, fall, winter}\}$ $|S| = 4$



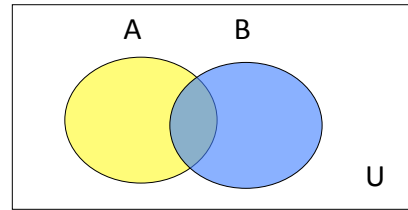
Example sets

- Alphabet
- All characters
- Booleans: true, false
- Numbers:
 - $N = \{0, 1, 2, 3, \dots\}$ Natural numbers
 - $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ Integers
 - $Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$ Rationals
 - R , Real Numbers
- Note that:
 - Q and R are not the same. Q is a *subset* of R .
 - N is a subset of Z .



Venn Diagram

- Graphical representation of set relations:



What is a set?

- Defining a set:
 - Option 1: List the members
 - Option 2; Use a set builder that defines set of x that hold a certain characteristic
 - Notation: $\{x \in S \mid \text{characteristic of } x\}$
 - Examples:
 - $A = \{x \in Z^+ \mid x \text{ is prime}\}$ – set of all prime positive integers
 - $O = \{x \in N \mid x \text{ is odd and } x < 10000\}$ – set of odd natural numbers less than 10000



Equality

- Two sets are *equal* if and only if (iff) they have the same elements.
- We write $A=B$ when for all elements x , x is a member of the set A iff x is also a member of B .
 - Notation: $\forall x \{x \in A \leftrightarrow x \in B\}$
 - For all values of x , x is an element of A if and only if x is an element of B



Set Operations

- Operations that take as input sets and have as output sets
- Operation1: *Union*
 - The union of the sets A and B is the set that contains those elements that are either in A or in B , or in both.
 - Notation: $A \cup B$
 - Example: union of $\{1,2,3\}$ and $\{1,3,5\}$ is?



Operation 2: Intersection

- The intersection of sets A and B is the set containing those elements in both A and B .
- Notation: $A \cap B$
- Example: $\{1,2,3\}$ intersection $\{1,3,5\}$ is?
- The sets are disjoint if their intersection produces the empty set.



Operation3: Difference

- The difference of A and B is the set containing those elements that are in A but not in B .
- Notation: $A - B$
- Aka the complement of B with respect to A
- Example: $\{1,2,3\}$ difference $\{1,3,5\}$ is?
- Can you define Difference using union, complement and intersection?



Operation3: Complement

- The complement of set A is the complement of A with respect to U , the universal set.
- Notation: \overline{A}
- Example: If \mathbb{N} is the universal set, what is the complement of $\{1,3,5\}$?
Answer: $\{0, 2, 4, 6, 7, 8, \dots\}$



Identities

Identity	$A \cup \emptyset = A, A \cap U = A$
Commutative	$A \cup B = B \cup A, A \cap B = B \cap A$
Associative	$A \cup (B \cap C) = (A \cup B) \cap C, A \cap (B \cup C) = (A \cap B) \cup C$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Complement	$A \cup \bar{A} = U, A \cap \bar{A} = \emptyset$



Subset

- The set A is said to be a subset of B iff for all elements x of A, x is also an element of B.
But not necessarily the reverse...
- Notation: $A \subseteq B \quad \forall x \{x \in A \rightarrow x \in B\}$
 - Unidirectional implication
- $\{1,2,3\} \subseteq \{1,2,3\}$
- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$
- What is the cardinality between sets if $A \subseteq B$?

Answer: $|A| \leq |B|$



Subset

- **Subset** is when a set is contained in another set. Notation: \subseteq
 - **Proper subset** is when A is a subset of B, but B is not a subset of A. Notation: \subset
 - $\forall x ((x \in A) \rightarrow (x \in B)) \wedge \exists x ((x \in B) \wedge (x \notin A))$
 - All values x in set A also exist in set B
 - ... but there is at least 1 value x in B that is not in A
 - $A = \{1,2,3\}, B = \{1,2,3,4,5\}$
- $A \subset B$, means that $|A| < |B|$.



Empty Set

- **Empty set** has no elements and therefore is the subset of all sets. $\{\}$ Alternate Notation: \emptyset
- Is $\emptyset \subseteq \{1,2,3\}$? - Yes!
- The cardinality of \emptyset is zero: $|\emptyset| = 0$.
- Consider the set containing the empty set: $\{\emptyset\}$.
- Yes, this is indeed a set: $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.



Set Theory - Definitions and notation

- Quiz time:
 - $A = \{x \in \mathbb{N} \mid x \leq 2000\}$ What is $|A| = 2001$?
 - $B = \{x \in \mathbb{N} \mid x \geq 2000\}$ What is $|B| =$ Infinite!
 - Is $\{x\} \subseteq \{x\}$? Yes
 - Is $\{x\} \in \{x, \{x\}\}$? Yes
 - Is $\{x\} \subseteq \{x, \{x\}\}$? Yes
 - Is $\{x\} \in \{x\}$? No



Powerset

- The powerset of a set is the set containing *all* the subsets of that set.
- Notation: $\mathbf{P}(A)$ is the powerset of set A.
- Fact: $|\mathbf{P}(A)| = 2^{|A|}$.
- If $A = \{x, y\}$, then $\mathbf{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$
- If $S = \{a, b, c\}$, what is $\mathbf{P}(S)$?



Powerset example

- Number of elements in powerset = 2^n where $n = \#$ elements in set
 - S is the set $\{a, b, c\}$, what are all the subsets of S ?
 - $\{\}$ – the empty set
 - $\{a\}, \{b\}, \{c\}$ – one element sets
 - $\{a, b\}, \{a, c\}, \{b, c\}$ – two element sets
 - $\{a, b, c\}$ – the original set
- and hence the power set of S has $2^3 = 8$ elements:

$\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$



Example

- Consider binary numbers
 - E.g. 0101
- Let every bit position $\{1, \dots, n\}$ be an item
 - Position i is in the set if bit i is 1
 - Position i is not in the set if bit i is 0
- What is the set of all possible N -bit numbers?
 - *The powerset of $\{1, \dots, n\}$.*



Why sets?

- Programming - Recall a *class*... it is the set of all its possible objects.
- We can restrict the *type* of an object, which is the set of values it can hold.
 - Example: Data Types
 - int set of integers (finite)
 - char set of characters (finite)
 - Is \mathbb{N} the same as the set of integers in a computer?



Order Matters

- What if order matters?
 - Sets disregard ordering of elements
 - If order is important, we use *tuples*
 - If order matters, then are duplicates important too?



Tuples

- Order matters
- Duplicates matter
- Represented with parens ()
- Examples
 - $(1, 2, 3) \neq (3, 2, 1) \neq (1, 1, 1, 2, 3, 3)$
 - (a_1, a_2, \dots, a_n)



Tuples

- The *ordered n -tuple* (a_1, a_2, \dots, a_n) is the ordered collection that has a_1 as its first element a_2 as its second element ... and a_n as its n th element.
- An *ordered pair* is a 2-tuple.
- Two ordered pairs (a,b) and (c,d) are equal iff $a=c$ and $b=d$ (e.g. *NOT* if $a=d$ and $b=c$).
- A 3-tuple is a *triple*; a 5-tuple is a *quintuple*.



Tuples

- In programming?
 - Let's say you're working with three integer values, first is the office room # of the employee, another is the # years they've worked for the company, and the last is their ID number.
 - Given the following set {320, 13, 4392}, how many years has the employee worked for the company?
 - What if the set was {320, 13, 4392}? Doesn't {320, 13, 4392} = {320, 4392, 13} ?
 - Given the 3-tuple (320, 13, 4392) can we identify the number of years the employee worked?



Why?

- Because ordered n-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position (2, 3) in a maze: **new Mouse (2, 3)**
- Can we reverse the order of the parameters?
- From Java, **Math.min (1, 2)**



Cartesian Product of Two Sets

- Let A and B be sets. The Cartesian Product of A and B is the set of all ordered pairs (a,b), where $b \in B$ and $a \in A$
- Cartesian Product is denoted $A \times B$.
- Example: $A = \{1, 2\}$ and $B = \{a, b, c\}$. What is $A \times B$ and $B \times A$?



Cartesian Product

- $A = \{a, b\}$
- $B = \{1, 2, 3\}$
- $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- $B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Functions in CS

- Function = mappings or transformations
- Examples

$$f(x) = x$$

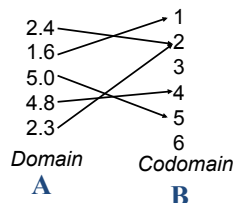
$$f(x) = x + 1$$

$$f(x) = 2x$$

$$f(x) = x^2$$

Function Definitions

- A function f from sets A to B assigns exactly one element of B to each element of A .
- Example: the **floor** function



What's the difference between codomain and range?

Range: {1,2,4,5}

Range contains the codomain values that A maps to

Function Definitions

- In Programming
 - Function header definition example

```
int floor( float real)
{
}

```

- Domain = \mathbb{R}
- Codomain = \mathbb{Z}

Other Functions

- The **identity** function, f_{ID} , on A is the function where: $f_{ID}(x) = x$ for all x in A .

$$A = \{a, b, c\} \text{ and } f(a) = a, f(b) = b, f(c) = c$$

- **Successor function**, $f_{succ}(x) = x+1$, on \mathbb{Z}

- $f(1) = 2$
- $f(-17) = -16$
- $f(a)$ Does NOT map to b

Only works on set \mathbb{Z}

- **Predecessor function**, $f_{pred}(x) = x-1$, on \mathbb{Z}

- $f(1) = 0$
- $f(-17) = -18$



Other Functions

- $f_{NEG}(x) = -x$, also on \mathbf{R} (or \mathbf{Z}), maps a value into the negative of itself.
- $f_{SQ}(x) = x^2$, maps a value, x , into its square, x^2 .
- The **ceiling** function: $ceil(2.4) = 3$.



Functions in CS

- What are ceiling and floor useful for?
 - Data stored on disk are represented as a string of bytes. Each byte = 8 bits. How many bytes are required to encode 100 bits of data?



Need smallest integer that is at least as large as $100/8$

$100/8 = 12.5$
But we don't work with $\frac{1}{2}$ a byte.
So we need 13 bytes



What is NOT a function?

- Consider $f_{SQRT}(x)$ from \mathbf{Z} to \mathbf{R} .
- This does **not** meet the given definition of a function, because $f_{SQRT}(16) = \pm 4$.
- In other words, $f_{SQRT}(x)$ assigns exactly one element of \mathbf{Z} to two elements of \mathbf{R} .



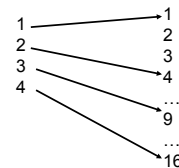
No Way!
Say it ain't so!!

Note that the convention is that \sqrt{x} is always the positive value.
 $f_{SQRT}(x) = \pm\sqrt{x}$



1 to 1 Functions

- A function f is said to be *one-to-one* or *injective* if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f .
- Example: the **square** function from \mathbf{Z}^+ to \mathbf{Z}^+





1 to 1 Functions, cont.

- Is **square** from \mathbb{Z} to \mathbb{Z} an example?
 - **NO!**
 - Because $f_{sq}(-2) = 4 = f_{sq}(+2)$!
- Is **floor** an example?
 - INCONCEIVABLE!!**
- Is **identity** an example?
 - Unique at last!!**



How dare they have the same codomain!



Increasing Functions

- A function f whose domain and co-domain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$ and *strictly increasing* if $f(x) < f(y)$, whenever
 - $x < y$ and
 - x and y are in the domain of f .

So YES floor is an increasing function

BUT it is NOT STRICTLY increasing

- Is **floor** an example?

$1.5 < 1.7$ and $\text{floor}(1.5) = 1 = \text{floor}(1.7)$
 $1.2 < 2.2$ and $\text{floor}(1.2) = 1 < 2 = \text{floor}(2.2)$

- Is **square** an example?

When mapping \mathbb{Z} to \mathbb{Z} or \mathbb{R} to \mathbb{R} :
 $\text{square}(-2) = 4 > 1 = \text{square}(1)$ yet $-2 < 1$

NO square is NOT an increasing function UNLESS...

Domain is restricted to positive #'s



How is Increasing Useful?

- Most programs run longer with larger or more complex inputs.
- Consider the maze:
 - Larger maze *may* (in the worst case) take longer to get out.
 - Maze with more walls *may* (in the worst case) take longer to get out.
- Consider looking up a telephone number in the paper directory...



Cartesian Products and Functions

- A function with multiple arguments maps a Cartesian product of inputs to a codomain.

- Example:

– **Math.min** maps $\mathbb{Z} \times \mathbb{Z}$ to \mathbb{Z}

`int minVal = Math.min(23, 99);`

– **Math.abs** maps \mathbb{Q} to \mathbb{Q}^+

`int absVal = Math.abs(-23);`

Find the minimum value between two integers

Find the absolute value of a number



Quiz Check

- Is the following an increasing function?

$$\mathbb{Z} \rightarrow \mathbb{Z} \quad f(x) = x + 5$$

$$\mathbb{Z} \rightarrow \mathbb{Z} \quad f(x) = 3x - 1$$