



## Propositional Logic, Truth Tables, and Predicate Logic (Rosen, Sections 1.1, 1.2, 1.3)

### TOPICS

- Propositional Logic
- Logical Operations
- Equivalences
- Predicate Logic



## Logic?



## What is logic?

Logic is a truth-preserving system of inference

**Truth-preserving:**  
If the initial statements are true, the inferred statements will be true

**System:** a set of mechanistic transformations, based on syntax alone

**Inference:** the process of deriving (inferring) new statements from old statements



## Propositional Logic

- A *proposition* is a statement that is either true or false
- Examples:
  - This class is CS160 (true)
  - Today is Sunday (false)
  - It is currently raining in Singapore (???)
- Every proposition is true or false, but its *truth value* (true or false) may be unknown



## Propositional Logic (II)

- A propositional statement is one of:
  - A simple proposition
    - denoted by a capital letter, e.g. 'A'.
  - A negation of a propositional statement
    - e.g.  $\neg A$  : "not A"
  - Two propositional statements joined by a *connective*
    - e.g.  $A \wedge B$  : "A and B"
    - e.g.  $A \vee B$  : "A or B"
  - If a connective joins complex statements, parenthesis are added
    - e.g.  $A \wedge (B \vee C)$

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## Truth Tables

- The truth value of a compound propositional statement is determined by its truth table
- Truth tables define the truth value of a connective for every possible truth value of its terms

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## Logical negation

- Negation of proposition A is  $\neg A$ 
  - A: It is snowing.
  - $\neg A$ : It is not snowing
  - A: Newton knew Einstein.
  - $\neg A$ : Newton did not know Einstein.
  - A: I am not registered for CS195.
  - $\neg A$ : I am registered for CS195.

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## Negation Truth Table

$A$	$\neg A$
0	1
1	0

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## Logical and (*conjunction*)

- Conjunction of A and B is  $A \wedge B$ 
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - $A \wedge B$ : CS160 teaches logic and Java.
- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi:  $A \wedge \neg B$



## Truth Table for Conjunction

$A$	$B$	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1



## Logical or (*disjunction*)

- Disjunction of A and B is  $A \vee B$ 
  - A: Today is Friday.
  - B: It is snowing.
  - $A \vee B$ : Today is Friday or it is snowing.
- This statement is true if any of the following hold:
  - Today is Friday
  - It is snowing
  - Both
- Otherwise it is false



## Truth Table for Disjunction

$A$	$B$	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1



## Exclusive Or

- The “or” connective  $\vee$  is inclusive: it is true if either *or both* arguments are true
- There is also an exclusive or  $\oplus$

$A$	$B$	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

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## Confusion over Inclusive OR and Exclusive OR

- Restaurants typically let you pick one (either soup or salad, not both) when they say “The entrée comes with a soup or salad”.
  - Use exclusive OR to write as a logic proposition
- Give two interpretations of the sentence using inclusive OR and exclusive OR:
  - Students who have taken calculus or intro to programming can take this class

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## Conditional & Biconditional Implication

- The conditional implication connective is  $\rightarrow$
- The biconditional implication connective is  $\leftrightarrow$
- These, too, are defined by truth tables

$A$	$B$	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

$A$	$B$	$A \leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

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## Conditional implication

- A: A programming homework is due.
- B: It is Tuesday.
- $A \rightarrow B$ :
  - If a programming homework is due, then it must be Tuesday.
  - A programming homework is due only if it is Tuesday.
- Is this the same?
  - If it is Tuesday, then a programming homework is due.

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## Bi-conditional

- A: You can drive a car.
- B: You have a driver's license.
- $A \leftrightarrow B$ 
  - You can drive a car if and only if you have a driver's license (and vice versa).
- What if we said "if"?
- What if we said "only if"?

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## Compound Truth Tables

- Truth tables can also be used to determine the truth values of compound statements, such as  $(A \vee B) \wedge (\neg A)$  (fill this as an exercise)

A	B	$\neg A$	$A \vee B$	$(A \vee B) \wedge (\neg A)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

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## Tautology and Contradiction

- A **tautology** is a compound proposition that is always true.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is neither a tautology nor a contradiction.
- A compound proposition is **satisfiable** if there is at least one assignment of truth values to the variables that makes the statement true.

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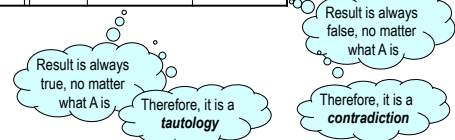
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## Examples

A	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$
0	1	1	0
1	0	1	0



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## Logical Equivalence

- Two compound propositions,  $p$  and  $q$ , are logically equivalent if  $p \leftrightarrow q$  is a tautology.
- Notation:  $p \equiv q$
- De Morgan's Laws:
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- How so? Let's build a truth table!

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## Prove $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$p$	$q$	$\neg p$	$\neg q$	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0



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## Show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$p$	$q$	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0



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## Other Equivalences

- Show  $p \rightarrow q \equiv \neg p \vee q$
- Show Distributive Law:
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

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Show  $p \rightarrow q \equiv \neg p \vee q$

$p$	$q$	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1



Show  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p$	$q$	$r$	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1



More Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption

See Rosen for more.



Equivalences with Conditionals and Biconditionals

- Conditionals
  - $p \rightarrow q \equiv \neg p \vee q$
  - $p \rightarrow q \equiv \neg q \rightarrow \neg p$
  - $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- Biconditionals
  - $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
  - $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
  - $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$



## Prove Biconditional Equivalence

p	q	$\neg q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$
0	0	1	1	0	0
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0



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## Converse, Contrapositive, Inverse

- The *converse* of an implication  $p \rightarrow q$  reverses the propositions:  $q \rightarrow p$
- The *inverse* of an implication  $p \rightarrow q$  inverts both propositions:  $\neg p \rightarrow \neg q$
- The *contrapositive* of an implication  $p \rightarrow q$  reverses and inverts:  $\neg q \rightarrow \neg p$   
*The converse and inverse are not logically equivalent to the original implication, but the contrapositive is, and may be easier to prove.*

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## Predicate Logic

- Some statements cannot be expressed in propositional logic, such as:
  - All men are mortal.
  - Some trees have needles.
  - $X > 3$ .
- Predicate logic can express these statements and make inferences on them.

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## Statements in Predicate Logic

$P(x,y)$

- Two parts:
  - A predicate P describes a relation or property.
  - Variables (x,y) can take arbitrary values from some domain.
- Still have two truth values for statements (T and F)
- When we assign values to x and y, then P has a truth value.

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## Example

- Let  $Q(x,y)$  denote “ $x=y+3$ ”.
  - What are truth values of:
    - $Q(1,2)$  ... false
    - $Q(3,0)$  ... true
- Let  $R(x,y)$  denote  $x$  beats  $y$  in Rock/Paper/Scissors with 2 players with following rules:
  - Rock smashes scissors, Scissors cuts paper, Paper covers rock.
  - What are the truth values of:
    - $R(\text{rock}, \text{paper})$  ... false
    - $R(\text{scissors}, \text{paper})$  ... true



## Quantifiers

- Quantification expresses the extent to which a predicate is true over a set of elements.
- Two forms:
  - Universal  $\forall$
  - Existential  $\exists$



## Universal Quantifier

- $P(x)$  is true for all values in the domain  $\forall x \in D, P(x)$
- For every  $x$  in  $D$ ,  $P(x)$  is true.
- An element  $x$  for which  $P(x)$  is false is called a *counterexample*.
- Given  $P(x)$  as “ $x+1 > x$ ” and the domain of  $R$ , what is the truth value of:

$$\forall x P(x) \quad \dots \text{true}$$



## Example

- Let  $P(x)$  be that  $x > 0$  and  $x$  is in domain of  $R$ .
- Give a counterexample for:  $\forall x P(x)$

$$x = -5$$



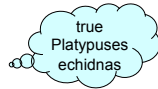
## Existential Quantifier

- $P(x)$  is true for at least one value in the domain.

$$\exists x \in D, P(x)$$

- For some  $x$  in  $D$ ,  $P(x)$  is true.
- Let the domain of  $x$  be “animals”,  
 $M(x)$  be “ $x$  is a mammal” and  
 $E(x)$  be “ $x$  lays eggs”,  
 what is the truth value of:

$$\exists x (M(x) \wedge E(x))$$



## English to Logic

- Some person in this class has visited the Grand Canyon.
- Domain of  $x$  is the set of all persons
- $C(x)$ :  $x$  is a person in this class
- $V(x)$ :  $x$  has visited the Grand Canyon
- $\exists x(C(x) \wedge V(x))$



## English to Logic

- For every one there is someone to love.
- Domain of  $x$  and  $y$  is the set of all persons
- $L(x, y)$ :  $x$  loves  $y$
- $\forall x \exists y L(x, y)$
- Is it necessary to explicitly include that  $x$  and  $y$  must be different people (i.e.  $x \neq y$ )?
  - Just because  $x$  and  $y$  are different variable names doesn't mean that they can't take the same values



## English to Logic

- No one in this class is wearing shorts and a ski parka.
- Domain of  $x$  is persons in this class
  - $S(x)$ :  $x$  is wearing shorts
  - $P(x)$ :  $x$  is wearing a ski parka
  - $\neg \exists x(S(x) \wedge P(x))$
- Domain of  $x$  is all persons
  - $C(x)$ :  $x$  belongs to the class
  - $\neg \exists x(C(x) \wedge S(x) \wedge P(x))$



## Evaluating Expressions: Precedence and Variable Bindings

- Precedence:
  - Quantifiers and negation are evaluated before operators
  - Otherwise left to right
- Bound:
  - Variables can be given specific values or
  - Can be constrained by quantifiers

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## Predicate Logic Equivalences

Statements are *logically equivalent* iff they have the same truth value under all possible bindings.

For example:

$$\forall x(P(x) \wedge Q(x)) \equiv \forall xP(x) \wedge \forall xQ(x)$$

In English: "Given the domain of students in CS160, all students have passed M124 course (P) and are registered at CSU (Q); hence, all students have passed M124 and all students are registered at CSU."

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## Other Equivalences

- Someone likes skiing (P) or likes swimming (Q); hence, there exists someone who likes skiing or there exists someone who likes swimming.

$$\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$$

- Not everyone likes to go to the dentist; hence there is someone who does not like to go to the dentist.

$$\neg \forall xP(x) \equiv \exists x\neg P(x)$$

- There does not exist someone who likes to go to the dentist; hence everyone does not like to go to the dentist.

$$\neg \exists xP(x) \equiv \forall x\neg P(x)$$

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