Rules of Inference
(Rosen, Section 1.6)

TOPICS

- Logic Proofs
  - ✧ via Truth Tables
  - ✧ via Inference Rules

Propositional Logic Proofs

- An *argument* is a sequence of propositions:
  - ✧ *Premises (Axioms)* are the first n propositions
  - ✧ *Conclusion* is the final proposition.
- An argument is *valid* if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology, given that p_i are the premises (axioms) and *q* is the conclusion.

Proof Method #1: Truth Table

- If the conclusion is true in the truth table whenever the premises are true, it is proved
  - Warning: when the premises are false, the conclusion my be true *or* false
- Problem: given *n* propositions, the truth table has 2^n rows
  - Proof by truth table quickly becomes infeasible

Example Proof by Truth Table

s = ((p v q) ∧ (¬p v r)) → (q v r)

p	*q*	*r*	¬p	p v q	¬p v r	q v r	(p v q)∧(¬p v r)	s
0	0	0	1	0	1	0	0	1
0	0	1	1	0	1	1	0	1
0	1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	0	0	1
1	0	1	0	1	1	1	1	1
1	1	0	0	1	0	1	0	1
1	1	1	0	1	1	1	1	1

Proof Method #2: Rules of Inference

- A *rule of inference* is a pre-proved relation: any time the left hand side (LHS) is true, the right hand side (RHS) is also true.
- Therefore, if we can match a premise to the LHS (by substituting propositions), we can assert the (substituted) RHS

Inference properties

- Inference rules are truth preserving
  - If the LHS is true, so is the RHS
- Applied to true statements
  - Axioms or statements proved from axioms
- Inference is syntactic
  - Substitute propositions
    - if p replaces q once, it replaces q everywhere
    - If p replaces q, it only replaces q
  - Apply rule

Example Rule of Inference

Modus Ponens

$$(p \wedge (p \to q)) \to q$$

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore\ q \end{array}$$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Rules of Inference

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \\ \underline{p \to q} \\ q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ p \to r \end{array}$$

Addition

$$\begin{array}{l} \underline{p} \\ p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ q \vee r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ q \end{array}$$

Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ \underline{q} \\ p \wedge q \end{array}$$

Logical Equivalences

Logical Equivalences

Idempotent Laws	DeMorgan's Laws	Distributive Laws
$p \vee p \equiv p$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge p \equiv p$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Double Negation	**Absorption Laws**	**Associative Laws**
$\neg(\neg p) \equiv p$	$p \vee (p \wedge q) \equiv p$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
	$p \wedge (p \vee q) \equiv p$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative Laws	**Implication Laws**	**Biconditional Laws**
$p \vee q \equiv q \vee p$	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \wedge q \equiv q \wedge p$	$p \rightarrow q \equiv \neg q \rightarrow \neg p$	$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$

Modus Ponens

- If p, and p implies q, then q

Example:

p = it is sunny, q = it is hot

$p \rightarrow q$, it is hot whenever it is sunny

"Given the above, if it is sunny, it must be hot".

Modus Tollens

- If not q and p implies q, then not p

Example:

p = it is sunny, q = it is hot

$p \rightarrow q$, it is hot whenever it is sunny

"Given the above, if it is not hot, it cannot be sunny."

Hypothetical Syllogism

- If p implies q, and q implies r, then p implies r

Example:

p = it is sunny, q = it is hot, r = it is dry

$p \rightarrow q$, it is hot when it is sunny

$q \rightarrow r$, it is dry when it is hot

"Given the above, it must be dry when it is sunny"

Disjunctive Syllogism

- If p or q, and not p, then q

Example:

p = it is sunny, q = it is hot

$p \vee q$, it is hot or sunny

“Given the above, if it not sunny, but it is hot or sunny, then it is hot”

Resolution

- If p or q, and not p or r, then q or r

Example:

p = it is sunny, q = it is hot, r = it is dry

$p \vee q$, it is sunny or hot

$\neg p \vee r$, it is not hot or dry

“Given the above, if it is sunny or hot, but not sunny or dry, it must be hot or dry”

Not obvious!

Addition

- If p then p or q

Example:

p = it is sunny, q = it is hot

$p \vee q$, it is hot or sunny

“Given the above, if it is sunny, it must be hot or sunny”

Of course!

Simplification

- If p and q, then p

Example:

p = it is sunny, q = it is hot

$p \wedge q$, it is hot and sunny

“Given the above, if it is hot and sunny, it must be hot”

Of course!

Conjunction

- If p and q, then p and q

Example:

p = it is sunny, q = it is hot

p ∧ q, it is hot and sunny

"Given the above, if it is sunny and it is hot, it must be hot and sunny"

Of course!

A Simple Proof

Given X, X→Y, Y →Z, ¬Z ∨ W, prove W

	Step	Reason
1.	$x \rightarrow y$	Premise
2.	$y \rightarrow z$	Premise
3.	$x \rightarrow z$	Hypothetical Syllogism (1, 2)
4.	x	Premise
5.	z	Modus Ponens (3, 4)
6.	$\neg z \vee w$	Premise
7.	w	Disjunctive Syllogism (5, 6)

A Simple Proof

"In order to sign up for CS161, I must complete CS160 and either M155 or M160. I have not completed M155 but I have completed CS161. Prove that I have completed M160."

STEP 1) Assign propositions to each statement.

- A : CS161
- B : CS160
- C : M155
- D : M160

Setup the proof

STEP 2) Extract axioms and conclusion.

- Axioms:
  - $A \rightarrow B \wedge (C \vee D)$
  - A
  - $\neg C$
- Conclusion:
  - D

Now do the Proof

STEP 3) Use inference rules to prove conclusion.

	Step	Reason
1.	$A \to B \wedge (C \vee D)$	Premise
2.	A	Premise
3.	$B \wedge (C \vee D)$	Modus Ponens (1, 2)
4.	$C \vee D$	Simplification (3)
5.	$\neg C$	Premise
6.	D	Disjunctive Syllogism (4, 5)

Another Example

Given:

$p \to q$

$\neg p \to r$

$r \to s$

Conclude:

$\neg q \to s$

Proof of Another Example

	Step	Reason
1.	$p \to q$	Premise
2.	$\neg q \to \neg p$	Implication law (1)
3.	$\neg p \to r$	Premise
4.	$\neg q \to r$	Hypothetical syllogism (2, 3)
5.	$r \to s$	Premise
6.	$\neg q \to s$	Hypothetical syllogism (4, 5)

Proof using Rules of Inference and Logical Equivalences

Prove: $\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge \neg q)$

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ — By 2nd DeMorgan's

$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ — By 1st DeMorgan's

$\equiv \neg p \wedge (p \vee \neg q)$ — By double negation

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ — By 2nd distributive

$\equiv F \vee (\neg p \wedge \neg q)$ — By definition of $\wedge$

$\equiv (\neg p \wedge \neg q) \vee F$ — By commutative law

$\equiv (\neg p \wedge \neg q)$ — By definition of $\vee$