Sets
(Rosen, Sections 2.1,2.2)

TOPICS

- Discrete math
- Set Definition
- Set Operations
- Tuples

Why Study Discrete Math?

- Digital computers are based on discrete units of data (bits).
- Therefore, both a computer's
  - structure (circuits) and
  - operations (execution of algorithms)

  can be described by discrete math
- A generally useful tool for rational thought! Prove your arguments.

What is 'discrete'?

- Consisting of distinct or unconnected elements, not continuous (calculus)
- Helps us in Computer Science:
  - What is the probability of winning the lottery?
  - How many valid Internet address are there?
  - How can we identify spam e-mail messages?
  - How many ways are there to choose a valid password on our computer system?
  - How many steps are needed to sort a list using a given method?
  - How can we prove our algorithm is more efficient than another?

Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, *etc.*…
- *i.e.*, the whole field!

What is a set?

- *An unordered collection of objects*
  - {1, 2, 3} = {3, 2, 1} since sets are unordered.
  - {a, b, c} = {b, c, a} = {c, b, a} = {c, a, b} = {a, c, b}
  - {2}
  - {on, off}
  - { }

What is a set?

- Objects are called *elements* or *members* of the set
- Notation ∈
  - a ∈ B means "a is an element of set B."
  - Lower case letters for elements in the set
  - Upper case letters for sets
  - If A = {1, 2, 3, 4, 5} and x ∈ A, what are the possible values of x?

What is a set?

- Infinite Sets *(without end, unending)*
  - N = {0, 1, 2, 3, …} is the Set of natural numbers
  - Z = {…, -2, -1, 0, 1, 2, …} is the Set of integers
  - Z+ = {1, 2, 3, …} is the Set of positive integers
- Finite Sets *(limited number of elements)*
  - V = {a, e, i, o, u} is the Set of vowels
  - O = {1, 3, 5, 7, 9} is the Set of odd #'s < 10
  - F = {a, 2, Fred, New Jersey}
  - Boolean data type used frequently in programming
    - B = {0,1}
    - B = {false, true}
  - Seasons = {spring, summer, fall, winter}
  - ClassLevel = {Freshman, Sophomore, Junior, Senior}

What is a set?

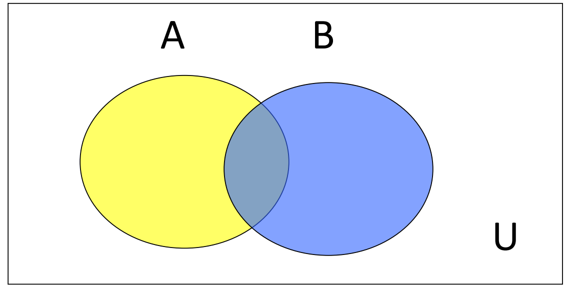
- Infinite vs. finite
  - If finite, then the number of elements is called the *cardinality*, denoted |S|
    - V = {a, e, i, o, u} |V| = 5
    - F = {1, 2, 3} |F| = 3
    - B = {0,1} |B| = 2
    - S = {spring, summer, fall, winter} |S| = 4

Example sets

- Alphabet
- All characters
- Booleans: true, false
- Numbers:
  - $N = \{0,1,2,3\ldots\}$ - Natural numbers
  - $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$ - Integers
  - $Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$ - Rationals
  - R, Real Numbers
- Note that:
  - Q and R are not the same. Q is a *subset* of R.
  - N is a subset of Z.

Venn Diagram

- Graphical representation of set relations:



What is a set?

- Defining a set:
  - Option 1: List the members
  - Option 2; Use a *set builder* that defines set of x that hold a certain characteristic
  - Notation: {x ∈ S | characteristic of x}
  - Examples:
    - A = { x ∈ Z^+ | x is prime } – set of all prime positive integers
    - O = { x ∈ N | x is odd and x < 10000 } – set of odd natural numbers less than 10000

Equality

- $A = B$ is used to show set equality
- Two sets are *equal* when they have exactly the same elements
- Thus for all elements x, x belongs to A *if and only if* (iff) x also belongs to B
- The if and only is a bidirectional implication that we will study later

Set Operations: Union

- Operations that take as input sets and have as output sets
- The *union* of the sets A and B is the set that contains those elements that are either in A or in B, or in both.
  - Notation: A ∪ B
  - Example: union of {1, 2, 3} and {1, 3, 5} is?

Answer: {1, 2, 3, 5}

Set Operations: Intersection

- The *intersection* of sets A and B is the set containing those elements in both A and B.
- Notation: A∩B
- The sets are disjoint if their intersection produces the empty set.
- Example: {1, 2, 3} intersection {1, 3, 5} is?

Answer: {1, 3}

Set Operations: Difference

- The *difference* of A and B is the set of elements that are in A but not in B.
- Notation: A - B
- Aka the complement of B with respect to A
- Can you define difference using union, complement and intersection?
- Example: {1, 2, 3} difference {1, 3, 5} is?

Answer: {2}

Set Operations: Complement

- The complement of set A is the complement of A with respect to U, the universal set.
- Notation: $\overline{A}$
- Example: If N is the universal set, what is the complement of {1, 3, 5}?

Answer: {0, 2, 4, 6, 7, 8, …}

Identities

Identity	$A \cup \varnothing = A, A \cap U = A$
Commutative	$A \cup B = B \cup A, A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$
Distributative	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Complement	$A \cup \overline{A} = U, A \cap \overline{A} = \varnothing$

Subsets

- The set A is a subset of B iff for all elements x of A, x is also an element of B. *But not necessarily the reverse…*
- Notation: A ⊆ B
- {1,2,3} ⊆ {1,2,3}
- {1,2,3} ⊆ {1,2,3,4,5}
- What is the relationship of the cardinality between sets if A ⊆ B ? |A| <= |B|

Subset

- Subset is when a set is contained in another set. Notation: ⊆
- Proper subset is when A is a subset of B, but B is not a subset of A. Notation: ⊂
  - $\forall x\, ((x \in A) \rightarrow (x \in B)) \land \exists x\, ((x \in B) \land (x \notin A))$
  - All values x in set A also exist in set B
  - … but there is at least 1 value x in B that is not in A
  - A = {1,2,3}, B = {1,2,3,4,5}

A ⊂ B, means that |A| < |B|.

except for infinite sets, e.g., N ⊂ Z, but |N| = |Z| = infinity

Empty Set

- Empty set has no elements and therefore is the subset of all sets: { } or ∅
- Is ∅ ⊆ {1,2,3}? - Yes!
- The cardinality of ∅ is zero: |∅| = 0.
- Consider the set containing the empty set: {∅}
- Yes, this is indeed a set:

  ∅ ∈ {∅} and ∅ ⊆ {∅}.

Set Theory

Quiz time:

- A = { x ∈ N | x ≤ 2000 } What is |A|? 2001
- B = { x ∈ N | x ≥ 2000 } What is |B|? Infinite
- Is {x} ⊆ {x}? Yes
- Is {x} ∈ {x,{x}}? Yes
- Is {x} ⊆ {x,{x}}? Yes
- Is {x} ∈ {x}? No

Powerset

- The *powerset* of a set is the set containing *all* the subsets of that set.
- Notation: $\mathcal{P}(A)$ is the powerset of set A.
- Fact: $|\mathcal{P}(A)| = 2^{|A|}$.
- If A = { x, y }, then ***P***(A) = {∅, {x}, {y}, {x,y} }
- If **S** = {a, b, c}, what is ***P***(**S**)?

Powerset example

- **Number of elements in powerset = 2^n where n = # elements in set**
- ***S*** is the set **{a, b, c},** what are all the subsets of ***S***?
  - { } – the empty set
  - {a}, {b}, {c} – one element sets
  - {a, b}, {a, c}, {b, c} – two element sets
  - {a, b, c} – the original set

and hence the power set of *S* has 2^3 = 8 elements:

{{}, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c}}

Example

- Consider binary numbers
  - E.g. 0101
- Let every bit position {1,…,n} be an item
  - Position *i* is in the set if bit *i* is 1
  - Position *i* is not in the set if bit *i* is 0
- What is the set of all possible n-bit numbers?
  - *The powerset of {1,…n}.*

Example (contd.)

- n= 4, i.e, 4 bits, each representing 1 item

1	2	3	4

0	0	0	0	⟶	{}, No item present
1	0	0	0	⟶	{1}, Item 1 present
0	1	0	0	⟶	{2}, Item 2 present
……					
1	1	0	0	⟶	{1, 2}, Items 1, 2 present
……					
1	1	1	0	⟶	{1, 2, 3}, Items 1, 2, 3 present
……					
1	1	1	1	⟶	{1, 2, 3, 4}, All items present

Why sets?

- Programming - Recall a *class*… it is the set of all its possible objects.
- We can restrict the *type* of an object, which is the set of values it can hold.
  - Example: Data Types

    int set of integers (finite)

    char set of characters (finite)
  - Is $\boldsymbol{N}$ the same as the set of integers in a computer?
  - Is $\boldsymbol{Q}$ or $\boldsymbol{R}$ the same as the set of doubles in a computer?

Order Matters

- What if order matters?
  - Sets disregard ordering of elements
  - If order is important, we use *tuples*
  - If order matters, then are duplicates important too?

Tuples

- Order matters
- Duplicates matter
- Represented with parens ( )
- Examples
  - (1, 2, 3) ≠ (3, 2, 1) ≠ (1, 1, 1, 2, 3, 3)

$$\left(a_1, a_2, \ldots, a_n\right)$$

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element a_2 as its second element … and a_n as its nth element.
- An *ordered pair* is a *2-tuple*.
- Two ordered pairs (a,b) and (c,d) are equal iff a=c and b=d *(e.g. NOT if a=d and b=c)*.
- A 3-tuple is a *triple*; a 5-tuple is a *quintuple*.

Tuples

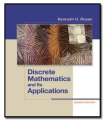
- In programming?
  - Let's say you're working with three integer values, first is the office room # of the employee, another is the # years they've worked for the company, and the last is their ID number.
    - Given the following <u>set</u> {320, 13, 4392}, how many years has the employee worked for the company?
    - What if the set was {320, 13, 4392}?
      Doesn't {320, 13, 4392} = {320, 4392, 13} ?
    - Given the <u>3-tuple</u> **(**320, 13, 4392**)** can we identify the number of years the employee worked?

Why?

- Because ordered n-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position (2, 3) in a maze: **`new Mouse(2,3)`**
- Can we reverse the order of the parameters?
- From Java, **`Math.min(1,2)`**

Cartesian Product

- Let A and B be sets. The Cartesian Product of A and B is the set of all ordered pairs (a,b), where b ∈ B and a ∈ A
- Cartesian Product is denoted A x B.
- Example: A={1,2} and B={a,b,c}. What is A x B and B x A?



Cartesian Product

- A = {a, b}
- B = {1, 2, 3}
- A X B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}
- B X A = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}