SOLUTIONS, Homework 6

1. Testing out the identities from Section 6.4.

   (a) Write out the numbers in the first nine rows of Pascal's triangle.

   ```
   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1
   1 5 10 10 5 1
   1 6 15 20 15 6 1
   1 7 21 35 35 21 7 1
   1 8 28 56 70 56 28 8 1
   ```

   (b) For each row i from 0 through 8, write out and add up the sum on the lefthand side of Corollary 1, to verify that it is equal to 2^i.

   ```
   1 = 2^0
   1 + 1 = 2^1
   1 + 2 + 1 = 4 = 2^2
   1 + 3 + 3 + 1 = 8 = 2^3, etc.
   ```

   (c) For each row i from 1 through 8, write out and add up the sum on the lefthand side of Corollary 2, to verify that it is equal to 0.

   ```
   1 - 1 = 0
   1 - 2 + 1 = 0
   1 - 3 + 3 - 1 = 0
   1 - 4 + 6 - 4 + 1 = 0, etc.
   ```

   (d) For each row i from 0 through 4, write out and add up the sum on the lefthand side of Corollary 3, and verify that it is equal to 3^i.

   ```
   2^0 * 1 = 1 = 3^0
   2^0 * 1 + 2^1 * 1 = 3 = 3^1
   2^0 * 1 + 2^1 * 2 + 2^2 * 1 = 9 = 3^2
   2^0 * 1 + 2^1 * 3 + 2^2 * 3 +  + 2^3 * 1 = 27 = 3^3, etc.
   ```

   (e) For row 4, add up the sum on the righthand side of Corollary 4, and verify that it adds up to the fourth element in row 8.

   ```
   1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = C(8,4)
   ```

2. Come up with a formula for the sum of elements in rows 0 through n of Pascal's triangle. *Hint: use the fact that each row i sums to 2^i, and use an identity about sums of powers of two that I proved in a class.*

   ```
   2^0 + 2^1 + 2^2 + ... + 2^n
   = 2^{n+1}-1.
   ```

3. In how many different ways can we select a president, vice-president, and secretary for our class if there are 60 people in it?

```
60*59*58 is a correct answer.  So is P(60,3).
```

4. In how many different ways can we select a committee of three people for our class if there are 60 people in it.

```
C(60,3)
```

5. How many bit strings of length 13 contain:

   (a) Exactly three 1's?

   ```
   C(13,3) = 286
   ```

   (b) At most three 1's?

   ```
   C(13,0) + C(13,1) + C(13,2) + C(13,3) = 1 + 13 + 78 + 286 = 378
   ```

   (c) At least four 1's? (Hint: your previous answer can be used to save you some arithmetic.)

   ```
   2^13 - 378 = 7814
   ```

   (d) At least three 1's?

   ```
   7814 + C(13,3) =  8100
   ```

6. In a group of n women and n men, there are $2(n!)^2$ ways to arrange them so that no two men are next to each other and no two women are next to each other. This is seen as follows: there are $n!$ ways to pick the order of women in the line and $n!$ ways to pick the order of men. By the product rule, there are $n!n!$ ways of choosing both of these orders. Once you have done this, there are two ways to interleave them to meet the constraints: one where a woman goes first and one where a man goes first. By the product rule, that's $2(n!)^2$

   Solve this problem: A group of people consists of n men and $2n$ women. How many ways are there to arrange these people in a row so that each woman is next to a woman and no woman is next to two women?

   ```
   n! ways to arrange the men
   (2n)! ways to arrange the women
   C(n+1,n) ways of inserting the n pairs of women in the n+1 places next to a man

   The answer is n!(2n)!C(n+1,n)
   ```

7. A dollar coin, a fifty-cent piece, a quarter, a dime, a nickel, and a penny are thrown up in the air.

(a) How many ways are there for the coins to come up heads and/or tails? Two ways that have the same number of heads are distinguishable if they don't all occur on the same coins.

```
2^6 = 64
```

(b) Notice that, since the coins are independent and each have a probability of 0.5 of coming up heads, none of these ways is more probable than any other. Since every coin must come up heads or tails, the sum of probabilities is 1.0.

What is the probability of each particular way?

```
1/64
```

(c) How many different ways are there for exactly three of the coins to come up heads?

```
C(6,3)
```

(d) What is the probability of getting exactly three heads?

```
C(6,3)*(1/64)
```

8. Find a formula for the number of ways to seat n people around a circular table:

(a) Find the number of ways to seat n people around the table and name one of them the chair of the meeting.

Note that you can match each such arrangement with a list of names, by going clockwise around the circle and starting with the chair. Similarly, each list of names is the match of one such arrangement, which is obtained by placing the list clockwise around the circle and naming the first person to be the chair. The listings and arrangements are all paired up, so the number of arrangements must be equal to the number of listings, for the same reason that if every man and woman at a dance has a dance partner, the number of men and women must be equal. At such a dance, you can find out how many men there are by counting the women.

```
There are n! listings.  Since these can be paired up with circular
arrangements that have a chair, there are n! circular arrangements with a
chair.
```

(b) Two ways to seat n ordinary people around a table are considered the same if one can be obtained from the other by rotating the table. In this situation, nobody has any special role. How many times does your answer to the previous part count each arrangment?

```
The above problem counted each arrangement n times, one for each choice
of the chair.
```

(c) Using your answer to the previous part, give a formula for how many ways there are to seat n ordinary people around the table. (Your answer should contain an n, an integer, an arithmetic operator, two parentheses, and a factorial symbol.)

```
n!/n = (n-1)!
```

9. Here is an identity: $C(n, k) = (n/k)C(n-1, k-1)$ for $0 < k \leq n$.

Here is one proof that it always works: $C(n, k) = n!/(k!(n-k)!) = n(n-1)!/(k(k-1)!(n-k)!) = (n/k)[(n-1)!/((k-1)!(n-k)!)] = (n/k)C(n-1, k-1)$.

That wasn't too hard, but you should practice coming up with combinatorial proofs for when the algebra is a mess. Here is a combinatorial proof of it. $C(n, k)$ is the number of ways to select a committee of size k from a set of n people, where nobody in the committee has any special status. Let's select the committee by selecting a *president* and the $k-1$ ordinary members, and then demote the president to an ordinary member to get a committee where nobody is special.

There are n ways to select the president. Once we have done that, there are $n-1$ people to choose from, and we must select $k-1$ of them to join the committee. There are $nC(n-1, k-1)$ to do this. After we demote the president, we see that each committee was counted k times, once for each way one of its k members could have gotten demoted from president. Dividing by the overcount of k, we get $(n/k)C(n-1, k-1)$.

Now, you must prove that $C(n, k) = n/(n-k)C(n-1, k)$ for $0 \leq k < n$ in two ways:

(a) Using the numbers from your Pascal's triangle, show that it works for $n = 8$ and $k = 3$.

```
C(8,3) = 56.  C(7,3) = 35.  35 * 8/5 = 56.
```

(b) Following the strategy of the first proof above, show it using algebra.

```
C(n,k) = n!/(k!(n-k)!) = n(n-1)!/(k!(n-k)(n-k-1)!) = (n/n-k)(n-1)!/(k!((n-1)-k)!
= n/(n-k)C(n-1,k)
```

(c) Show it using a combinatorial proof. Instead of choosing a president, choose a *loser* that you don't want to be on the committee. Give an expression for the number of ways to choose a committee without choosing the loser. Give an expression for the number of times each committee of size k got counted. Then use this to derive the formula.

```
n choices for the loser
C(n-1,k) ways to pick the committee once the loser has been excluded
We're up to nC(n-1,k) ways to pick a committee and a loser not on the
committee.

Each committee gets counted once for each choice of loser not on
the committee, so (n-k) times.  Dividing by this overcount factor,
we get n/(n-k)C(n-1,k)
```

10. Here is a way to write $10 + 12 + 14 + 16 + 18$ in Sigma notation: $\sum_{i=5}^{9} 2i$. Write the sum of the odd integers between 20 and 100 in Sigma notation.

    $\sum_{i=10}^{49}(2i+1)$ *is one way.* $\sum_{i=11}^{50}(2i-1)$ *is another.*

11. Using Pascal's triangle and the binomial theorem, find the expansion of $(x+y)^8$.

    $1x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + 1y^8$.

12. Write $P(10,3)$ as a product of consecutive integers, then give it as a single integer.

```
10*9*8 = 720
```

13. Write $C(10,3)$ as a ratio of $P(10,7)$ and another integer.

```
C(10,3) = C(10,7) = P(10,7)/7!
```

14. How many ways are there for first-, second-, and third-place prizes to be awarded to three people from a set of ten candidates?

```
P(10,3)
```

15. How many ways are there to give a \$100 award to three people from a set of ten candidates?

```
C(10,3)
```

16. In how many ways can you select a set of three *positive* integers *less than* 60?

```
C(59,3)
```

17. How imany odd-sized subsets are there in a set of size 100? *Hint: Use* $\sum_{k=0}^{n} C(n,k) = 2^n$ *and* $\sum_{k=0}^{n}(-1)^k C(n,k) = 0$ *for* $n > 0$.

```
There are 2^100 subsets by the first identity.  Exactly half of these have odd
size by the second identity.  So the answer must be 2^100/2 = 2^99
```

18. The English alphabet contains 26 letters. Twenty-one of them are consonants and five are vowels. How many strings of seven lowercase letters from the English alphabet are there?

```
26^7
```

19. How many of these don't have any repeated letters?

```
26*25*24*23*22*21*20 = P(26,7)
```

20. Here is a way to calculate the number of strings of lowercase letters of length 7 that don't have any repeated letters and have exactly two vowels: There are $C(21,5)$ ways to choose the five consonants and 5! ways to choose their order, for a total of $C(21,5) * 5! = P(21,5)$. Similarly, there are $P(5,2)$ ways to choose the two vowels and their order. There are $C(7,2)$ ways to choose the positions where these vowels go in the string of length 7. Each way to make these choices results in a different string, and each of the strings we want to count can be obtained by making these choices. The answer is $P(21,5)P(5,2)C(7,2)$.

    How many strings of lowercase letters of length 7 are there there that don't have any repeated letters and have exactly three vowels?

    ```
    P(21,4)P(5,3)C(7,3)
    ```

21. How many strings of lowercase letters of length 7 are there that don't have any repeated letters, have exactly three vowels, and end with the letter a? *Hint: This is the same as the number of strings of length 6 that don't have any repeated letters, have two vowels, and don't contain an a. We can match each such string with one of the ones we want to count by adding an a to the end, and match the ones we want to count with one of these by removing the final a. Every string in each set has exactly one partner in the other set.*

    Corrected solution:

    ```
    P(21,4)P(5,2)C(6,2)
    ```

22. How many strings of lowercase letters of length 7 are there that have exactly three vowels if repeated letters are allowed? *Hint: Choose a string of four consonants, choose a string of three vowels, and choose a way to interleave these.*

    ```
    21^4*5^3*C(7,3)
    ```