**Homework 8 written, CS161, Fall '11, McConnell**

**Due 12/9 in class.**

**Modified 12/9 13:53**

1. **Reading**: Some people have said that reading the Rosen text is painful. Let me show you how to make the first section of the chapter on induction in Rosen easy to read. There will be a graded quiz on Friday, 12/9, that will count toward your quiz grades, and at least one final exam question, each on examples from this section. You can ace them with certainty by following this short reading exercise.

   The strategy for induction is to show that a claim is true for the smallest case and then show that it can't have a smallest case where it's false.

   If you just glance through the examples, you will see Rosen's recipe emerging for the induction step.

   (a) $1 + 2 + 3 + ... + n = n(n + 1)/2$
   - Assume: $1 + 2 + 3 + ... + k = k(k + 1)/2$
   - Show: $1 + 2 + ...k + (k + 1) = (k + 1)(k + 2)/2$

   (b) $1 + 3 + 5 + ... + (2n - 1) = n^2$
   - Assume: $1 + 3 + 5 + ... + (2k - 1) = k^2$
   - Show: $1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = (k + 1)^2$

   In each case, he is just expressing the following, in terms of the problem at hand:
   - Assume: it's true for $k$.
   - Show: it's true for $k + 1$

   This shows that $k + 1$ can't be a smallest case where it's false.

   (I have often called these two cases $k - 1$ and $k$ instead of $k$ and $k + 1$. It makes no difference whether you say, "Bob is one year older than Frank," or "Frank is one year younger than Bob.")

   (c) Here's the next one: $1 + 2 + 4 + 8 + ... + 2^n = 2^{n+1} - 1$

   **Stop!** Put the book aside and see if you can guess what's coming next:
   - Assume:
   - Show:

   Then check whether you've guessed right.

   Now, go back through the examples and see how he gets from his assumption to what he's showing:
   - Assume: $1 + 2 + 3 + ... + k = k(k + 1)/2$
   - Show: $1 + 2 + ...k + (k + 1) = (k + 1)(k + 2)/2$

   He groups the lefthand side so he can apply the assumption: $[1 + 2 + ... + k] + (k + 1) = [k(k + 1)/2] + (k + 1)$. Then you see that that he does some algebra to show that this is $(k + 1)(k + 2)/2$.

   **Stop!** Put the book aside and see if you can do the algebra. Peek only if you're stumped.

   Look at the next one again:

- Assume: $1 + 3 + 5 + ... + (2k - 1) = k^2$
- Show: $1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = (k + 1)^2$

He groups the lefthand side so he can apply the assumption: $[1 + 3 + 5 + ... + (2k - 1)] + (2k + 1) = [k^2] + (2k + 1)$. Then you see that he then does some algebra to show that this is $(k + 1)^2$.

**Stop!** Put the book aside and see if you can show this using algebra. Peek only if you're stumped.

Look at the next one again:

- Assume: $1 + 2 + 4 + ... + 2^k = 2^{k+1} - 1$
- Show: $1 + 2 + 4 + 2^k + 2^{k+1} = 2^{k+2} - 1$.

**Stop!** See if you guess what's coming next with the grouping and rewriting. Then see if you guessed right. **Stop!** See if you can do the algebra without peeking.

Now let me point out that figuring these things out was probably a lot more interesting than passively letting Rosen take you through every last step. Moreover, you now have a good idea of what Rosen is going to say in all the text you've skipped over without even looking at it. Verify that you haven't overlooked anything by reading the skipped-over text. This is painless compared to reading all that text without already having a good idea of where it's going.

Work through the next few examples, about proving inequalities, doing nothing other than guessing what statement is assumed and what statement needs to be shown. Then go back through them and see if you can show why one follows from the other, looking in the book only when you're stumped. Do the same for the ones about divisibility results.

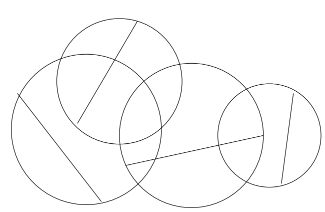
2. Recall the definition of Fibonacci numbers: $F(1) = 1$; $F(2) = 1$; $F(n) = F(n - 1) + F(n - 2)$ for all $n > 2$.

   By spending many hours staring at the Fibonacci numbers, Thad Johnston from our class has discovered a large number of identities involving them. Here is one: If $n$ is an even positive number, $F(n) = F(n - 1) + F(n - 3) + F(n - 5) + ... + F(1)$. In other words, $F(2m) = \sum_{i=1}^{m} F(2i - 1)$.

   For example, $F(8) = 21$ and $F(7) + F(5) + F(3) + F(1) = 1 + 2 + 5 + 13 = 21$.

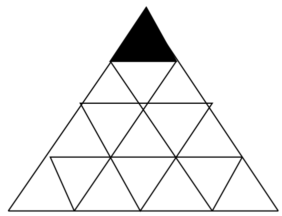
   Prove it by induction.

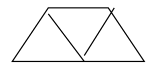
3. Show that if you place circles, each with a chord, in the plane, then the regions they form can be colored with three colors so that no touching regions have the same color. (To "touch," the regions have to lie on opposite sides of some segment of a curve, not just touch at a point.)



**Addendum: some people didn't realize that the infinite region is one of the regions that have to be colored. It is similar to the problem I solved in class about coloring the regions formed by a set of lines with two colors; some of those regions are unbounded.**

4. Here is a variation on the problem of covering a chessboard with widgets. In this problem you have an equilateral triangle consisting of smaller equilateral triangles whose sides are of unit length. The length of a side of the larger triangles is a power of two. Someone has removed one of the unit triangles at a corner of the larger triangle:





A *widget* is obtained by taking one of these boards made from four unit triangles and removing one. Show that the board can be tiled with widgets. The key element we're looking for is a figure, similar to Figure 8 of Section 4.2 of Rosen, that makes it suddenly obvious to anybody who understands induction.

5. **Typo corrected here:** Because $C(n,0) + C(n,1) + ...C(n,i) = \sum_{i=0}^{j} C(n,i)$ is a common expression in many practical problems, let's give it a name, such as $D(n,j)$.

   (a) Write an expression for the probability of getting at least $j$ heads in $n$ coin flips. Your expression should involve a power of two and $D(n,j)$.

   (b) Here's an idea for computing values of $D(n,j)$ for different values of $n$ and $j$. Start out just like you do with Pascal's triangle, except put powers of two on the diagonal instead of 1's.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 1 | | 4 | | | | |
| 3 | 1 | | | 8 | | | |
| 4 | 1 | | | | 16 | | |
| 5 | 1 | | | | | 32 | |
| 6 | 1 | | | | | | 64 |

   Then fill in the table using the same algorithm you use to fill in Pascal's triangle:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 1 | 3 | 4 | | | | |
| 3 | 1 | 4 | 7 | 8 | | | |
| 4 | 1 | 5 | 11 | 15 | 16 | | |
| 5 | 1 | 6 | 16 | 26 | 31 | 32 | |
| 6 | 1 | 7 | 22 | 42 | 57 | 63 | 64 |

Notice that $D(6,4) = C(6,0) + C(6,1) + C(6,2) + C(6,3) + C(6,4) = 1 + 6 + 15 + 20 + 15 = 57$ appears in row 6, column 4 of the table. Try out some other examples.

Will it always work? Try to prove it by induction. For the base cases, we have already seen that $D(n,0) = 1$ and $D(n,n) = 2^n$. That takes care of the numbers in the first column and on the diagonal. To finish out the proof, you just need to show that $D(n,j) = D(n-1,j) + D(n-1,j-1)$ when $0 < j < n$, which justifies using the same inductive procedure that you use for filling in Pascal's triangle.

6. Extra Credit. Thad has also noticed the following identity: $F(2n) = (F(n+1))^2 - (F(n-1))^2$ for $n \geq 2$. See if you can prove it. *Hint: Induction will probably fail you so try another trick you've learned: there is a nice combinatorial proof based on the number of ways to park Civics and Excursions. As a warmup, show that* $F(2n+1) = (F(n+1))^2 + (F(n))^2$ *for* $n \geq 1$ *using a combinatorial argument about parking Civics and Excursions.*