Counting continued

Rosen, Chapter 6

Generalized pigeon hole principle

- There are 10 pigeons and 3 holes, what can we say?

Generalized pigeon hole principle

- There are 10 pigeons and 3 holes, what can we say?

  at least one hole has at least 4 pigeons

- N objects placed in k boxes, then

  at least one box has at least $\lceil N/k \rceil$ objects

Examples

- 100 people, at least how many are born in the same month?

- What is the minimum # students such that 6 get the same grade? (A,B,C,D,F)

ask yourself:

what are the pigeons, what are the holes

Permutations

- In a family of 5, how many ways can we arrange the members of the family in a line for a photograph?

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4

- How many permutations of n objects are there?

How many permutations

- How many permutations of n objects are there?
- Using the product rule:

$$n \cdot (n-1) \cdot (n-2) ,\ldots, 2 \cdot 1 = n!$$

The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation $a_1,\ldots,a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities i and j

An optimal TSP tour through Germany's 15 largest cities

Solving TSP

- Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.
- Need a method for generating all permutations
- Note: how many solutions to a TSP problem with n cities?

Generating Permutations

- Let's design a recursive algorithm that starts with permutation [0,1,2,3,..,n-1]
  - which elements should be placed in position 0?
  - what needs to be done next?
  - what is the base case?
- Let's write the program....

r-permutations

- An ordered arrangement of r elements of a set:

  r-permutations of a set with n elements: **$P(n,r)$**
- Example: List the 2-permutations of {a,b,c}.

  $P(3,2) = 3 \times 2 = 6$

- Let n and r be integers such that $0 \leq r \leq n$ then there are

  $$P(n,r) = n\,(n-1)\ldots(n-r+1)$$

  r-permutations of a set with n elements.

$$P(n, r) = n! / (n-r)!$$

r-permutations - example

- How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

Combinations

- How many poker hands (five cards) can be dealt from a deck of 52 cards?
- How is this different than r-permutations?

Combinations

- The number of r-combinations out of a set with n elements: C(n,r) also denoted as: $\binom{n}{k}$
- Example: {1,3,4} is a 3-combination of {1,2,3,4}
- Example: How many 2-combinations of {a,b,c,d}

r-combinations

- How many r-combinations?

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

- Notice: $C(n, r) = C(n, n-r)$
  - We can prove that without using the formula

Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.
- Ordered selections: **r-permutations**.
- Two unordered selections are the same if
  - the elements chosen are the same.

    (regardless of the order in which the elements are chosen)
- Unordered selections: **r-combinations.**

Relationship between P(n,r) and C(n,r)

- Suppose we want to compute P(n,r) .
- Constructing an r-permutation from a set of n elements can be thought as a 2-step process:

  Step 1: Choose a subset of r elements;

  Step 2: Choose an ordering of the r-element subset.
- Step 1 can be done in C(n,r) different ways.
- Step 2 can be done in r! different ways.
- Based on the multiplication rule, P(n,r) = C(n,r) · r!
- Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

r-combinations

- Example: How many bit strings of length n contain exactly r ones?

- Count the r-combinations for r from 0 to n
  - What do you get?
  - Does that make sense?

Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process.
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.
- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets
- Apply the inclusion/exclusion rule if
  - It is simple to over-count and then to subtract duplicates

Some more advice about Counting

- Make sure that

  1) every element is counted;

  2) no element is counted more than once.

  (avoid double counting)

- When using the addition rule:

  1) every outcome should be in some subset;

  2) the subsets should be disjoint; if they are not, subtract the overlaps

Example using Inclusion/Exclusion Rule

- Question: How many integers from 1 through 100 are multiples of 4 or multiples of 7 ?
  - Solution: Let A be the set of integers from 1 through 100 which are multiples of 4; B be the set of integers from 1 through 100 which are multiples of 7.
  - $A \cap B$ is the set of integers from 1 through 100 which are multiples of 4 and 7 hence multiples of 28.
  - We want to find $|A \cup B|$.