Part 1.
Recursion as a Problem-Solving Technique

CS 200 Algorithms and Data Structures

Outline

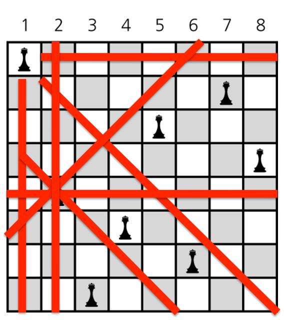
- Backtracking
- Formal grammars
- Relationship between recursion and mathematical induction

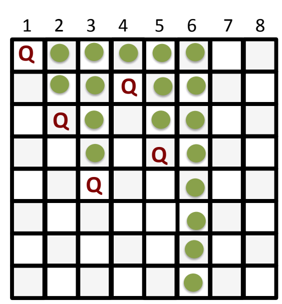
Backtracking

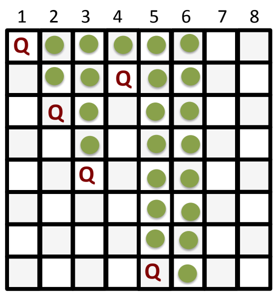
- Problem solving technique that involves **guesses** at a solution.
- Retrace steps in reverse order and try new sequence of steps

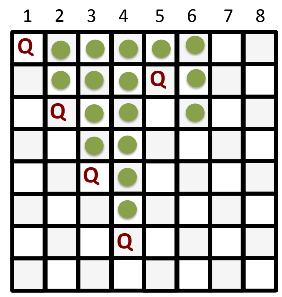
The Eight Queens Problem

Place 8 Queens!
- No queen can attack any other queens.









Solution with recursion and backtracking

```
placeQueen (in currColumn:integer)
if ( currColumn > 8) {
  The problem is solved
} else {
   while (unconsidered squares exist in currColumn and the
          problem is unsolved) {
       Determine if the next square is safe.
       if (such a square exists){
          place a queen in the square
          placeQueens(currColumn+1) // try next column
          if (no queen safe in currColumn+1) {
              remove queen from currColumn and try the next
               square in that col.
          }
       }
    }
}
```


Outline

- Backtracking
- Formal grammars
- Relationship between recursion and mathematical induction

Defining Languages

- Language: A set of strings of symbols from a finite alphabet.
- JavaPrograms = {strings *w*: *w* is a syntactically correct Java program}
- Grammar: the rules of a language
  - Determine whether a given string is in the language
  - Language Specifications

Some special symbols

- *x*|*y* means *x* or *y*
- *x y* means *x* followed by *y*
- <*word*> means any instance of word that the definition defines

Example

- Consider the language that the following grammar defines:
- < *S* > = % | < *W* > | %< *S* >
- < *W* > = *xy*|*x* < *W* > *y*
- Write all strings that are in this language

Example: Java Identifier

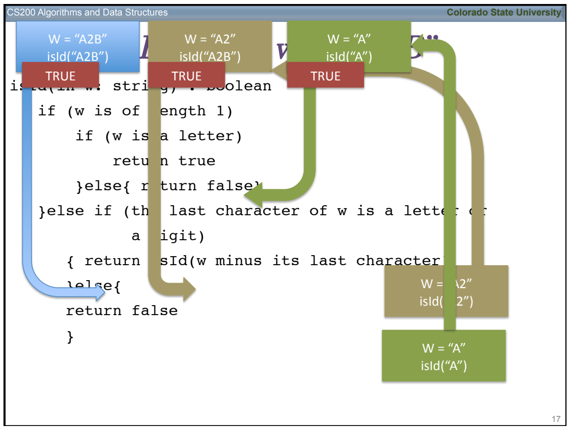
- A grammar for the language
  – JavaIds = {*w*: *w* is a legal Java identifier}
- Java identifiers are the names of variables, methods, classes, packages and interfaces
  – *Identifier: IdentifierChars but not a Keyword or BooleanLiterals or NullLiteral*
  – *IdentifierChars: JavaLetter or IdentifierChar or JavaLetterOrDigit*
  – *JavaLetter: any Unicode Character that is JavaLetter*
  – *JavaDigit: the ASCII digits 0-9*
  – *JavaLetterOrDigit: any Unicode Character that is JavaLetterOrDigit*
  – ***http://java.sun.com/docs/books/jls/download/langspec-3.0.pdf***

A Grammar for the Java Identifier

- *<identifierChars>* = *<JavaLetter>*| *<identifierChars><JavaLetter>*| *<identifierChars><JavaDigit>*|$<identifier>| _*<identifier>*
- *<letter>* = a|b|…|z|A|B|…|Z
- *<digit>* = *0|1|…|9*
- *An identifier is a letter, or an identifier followed by a letter, or an identifier followed by a digit.*

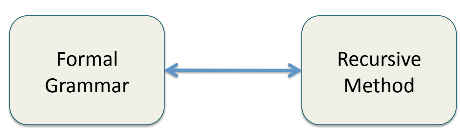
Recognition of JavaId

```
isId(in w: string): boolean
   if (w is of length 1) {
       if (w is a letter or $ or _) {
           return true
       } else {
          return false
       }
   } else if (the last character of w is a letter or a digit) {
         return isId(w minus its last character)
   } else {
         return false
   }
```

Grammar and recursive implementation of Palindromes

- Goal



- A *palindrome* is a string that reads the same from left to right as it does from right to left.

Find a Rule to satisfy all the Palindromes

- Examples: RADAR, RACECAR, MADAM, [A nut for a jar of Tuna]
- A *palindrome* is a string that reads the same from left to right as it does from right to left.
- Palindromes = {*w*: *w* reads the same left to right as right to left}
- If *w* is a palindrome
  - Then *w* minus its first and last characters is also a palindrome

More specifically

- The first and last characters of *w* are the same

  AND

- *w* minus its first and last characters is a palindrome

Base cases

- *Empty* string is palindrome
- A string of length **1** is a palindrome

Grammar for the language Palindrome

- *<pal> = empty string* | *<ch>*| *a <pal>a*|*b<pal>b*| …|*Z<pal>Z*
- *<ch>* = *a*|*b*|..*z*|*A*|*B*|…|*Z*

Example isPal ("RADAR") — TRUE

isPal ("ADA") — TRUE

isPal ("D") — TRUE

```
isPal(in w:string):boolean
  if (w is an empty string or of length 1) {
     return true
  } else if (w's first and last characters are the same) {
        return isPal(w minus its first and last
                     characters)
  } else {
        return false
  }
```

isPal ("ADA")

isPal ("D")

Algebraic Expressions

- Infix
  - Every binary operator appears between its operands

    a + b, *a+(b*c)*, *(a+b)*c*
- Prefix
  - Operator appears before its operands

    + a b, + *a* * *b c*, * + *a b c*
- Postfix
  - Operator appears after its operands

    a b +, *a b c* * +, *a b* + *c* *

Examples

Question 1) - x 3 8 + 6 5

Question 2) + -5 2 x 10 2

Question 3) 3 8 x 6 5 + -

Question 4) 5 2 – 10 2 x +

Prefix Expressions

<prefix> = *<identifier>*|*<operator><prefix><prefix>*

<operator>= +| - | *| /

<identifier>= *a*|*b*|…|*z*

Recognize Prefix expressions

- Is the first character of input string an operator?
- Does the remainder of input string consist of two consecutive prefix expressions?

Recognize the end of prefix expressions

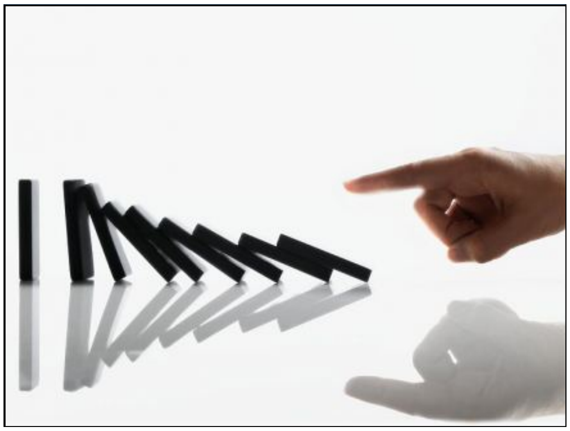
```
endPre (in first:integer, in last:integer):integer
 if (first < 0 or first > last){return -1}// noprefix
ch = character at position first of strExp
  if (ch is identifier){ return first }
     else if { ch is an operator} {
         firstEnd = endPre(first +1, last)
         if (firstEnd > -1) {
            return endPre(firstEnd +1, last)
         } else {
            return -1
          }
   }else {
     return -1
   }
```


Example

- Trace of *endPre (first, last)*, where *strExp* is *+/ ab-cd*

Outline

- Backtracking
- Formal grammars
- **Relationship between recursion and mathematical induction**



Mathematical Induction in Dominos

- We have N dominos.
- **If we push the 1st domino, will N dominos fall?**
  - We should show:
    - If we push *the 1st one*, it falls
    - For all of dominos, if the previous domino falls, next domino falls
- Process:
  - Show something works the first time
  - Assume that it works for this time
  - Show it will work for the next time, under the assumption
  - Conclusion, it works all the time

Principle of Mathematical Induction

- To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function,
- Two parts of mathematical induction
  - **Basis step**: verify that $P(1)$ is true
  - **Inductive step**: Show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all (positive, or non-negative) integers k.
- P(n): Propositional function
- P(k): Inductive hypothesis

Example

- Use mathematical induction to show that, 1+2+3+ … + n = n(n+1)/2

for all positive integer n.

Question 1. What is the propositional function here?

Question 2. What is the inductive hypothesis?

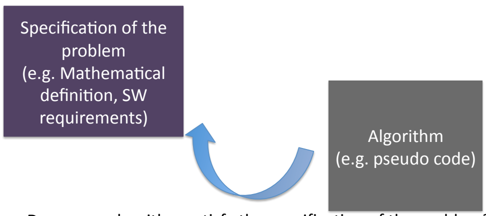
Recursion

- Specifies a solution to one or more base cases
- Then demonstrates how to derive the solution to a problem of an arbitrary size
  - From the smaller size of the same problem.

Mathematical Induction

- Proves a property about the natural numbers by
  - Proving the property about a base case and
  - Then proving that the property must be true for an arbitrary natural N if it is true for the natural number smaller than N.
- In this section, we will use MI to prove:
  - **(1) correctness of the recursive algorithm**
  - **(2) deriving the amount of recursive work it requires**

(1) Correctness of the Recursive Factorial Method



Does your algorithm satisfy the specification of the problem?

(1) Correctness of the Recursive Factorial Method

Definition of Factorial

$factorial(n) = n\ (n-1)\ (n-2)\ \ldots\ 1$ for any integer $n > 0$

$factorial(0) = 1$

Definition of method *fact(N)*

```
fact (in n: integer): integer
    if (n is 0) {
       return 1
    } else {
       return n* fact(n-1)
    }
```


Prove that the method fact computes the factorial of its arguments

Basis step:

: $fact(0) = 1$

Inductive Step:

Show that for an arbitrary positive integer k, if $fact(k)$ returns $k!$, $fact(k+1)$ returns $(k+1)!$

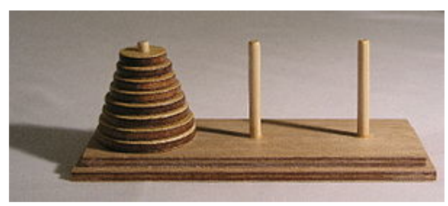
Assume that, $fact(k) = k\ (k-1)\ (k-2)\ \ldots\ 2\ 1$

For $n = k+1$,

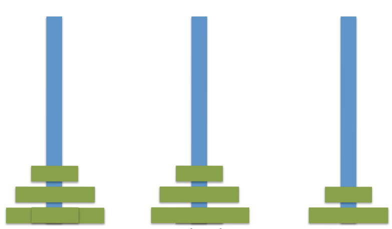
Show that $fact(k+1)$ returns $(k+1)\ k\ (k-1)\ (k-2)\ \ldots\ 2\ 1$

(2) Deriving the amount of recursive work

- **The Towers of Hanoi Example**
- **Only one** disk may be moved at a time.
- No disk may be placed on top of a smaller disk.



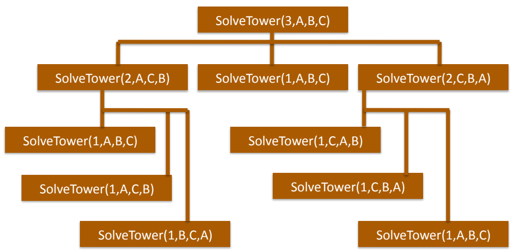
States in the Towers of Hanoi



Recursive Solution

```
solveTowers (in count: integer, in source: Pole, in
  destination: Pole, in spare:Pole)
   if (count is 1) {
       Move a disk directly from source to destination
   } else{
       solveTowers(count-1, source, spare, destination)
       solveTowers(1, source, destination, spare)
       solveTowers(count-1, spare, destination, source)
   }
```

Example with 3 disks



Cost of Towers of Hanoi

- If we have N disks, how many moves does *solveTowers()* make to solve the problem?
- From the software

  $moves(1) = 1$

  $move(N) = move(N\text{-}1)+1+move(N\text{-}1)$ (if $N>1$)
- A closed form formula for the number of moves that solveTowers requires for N disks:

  $moves(N) = 2^N - 1$ (for all $N>=1$)
- **Is this true for the *solveTowers()* method with N disks?**

Proof

- Basis Step
  - Show that the property is true for $N = 1$.

    $2^1 - 1 = 1$, which is consistent with the recurrence relation's specification that $moves(1) = 1$
- Inductive Step
  - Property is true for an arbitrary k ➔ property is true for $k+1$
  - Assume that the property is true for $N = k$

    $moves(k) = 2^k\text{-}1$
  - Show that the property is true for $N = k + 1$

Proof – cont.

- $moves(k+1) = 2 * moves(k) + 1$

  $= 2 * (2^k - 1) + 1$

  $= 2^{k+1} - 1$

Therefore the inductive proof is complete.

Readings for next class

- Stacks