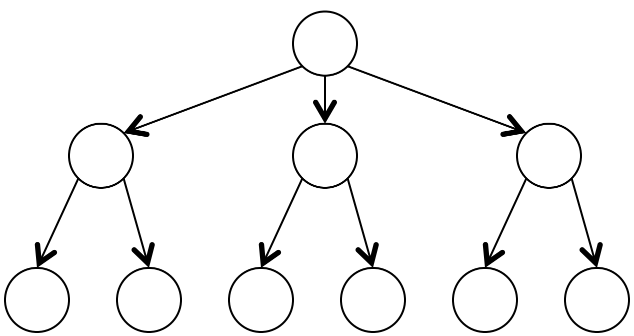
CS200: Recursion and induction (recap from cs161)

Prichard Ch. 6.1 & 6.3

Backtracking

- Problem solving technique that involves **moves: guesses** at a solution.
- **Depth First Search:** keep making guesses; in case of failure retrace steps and try a new move in a state with still unexplored guesses

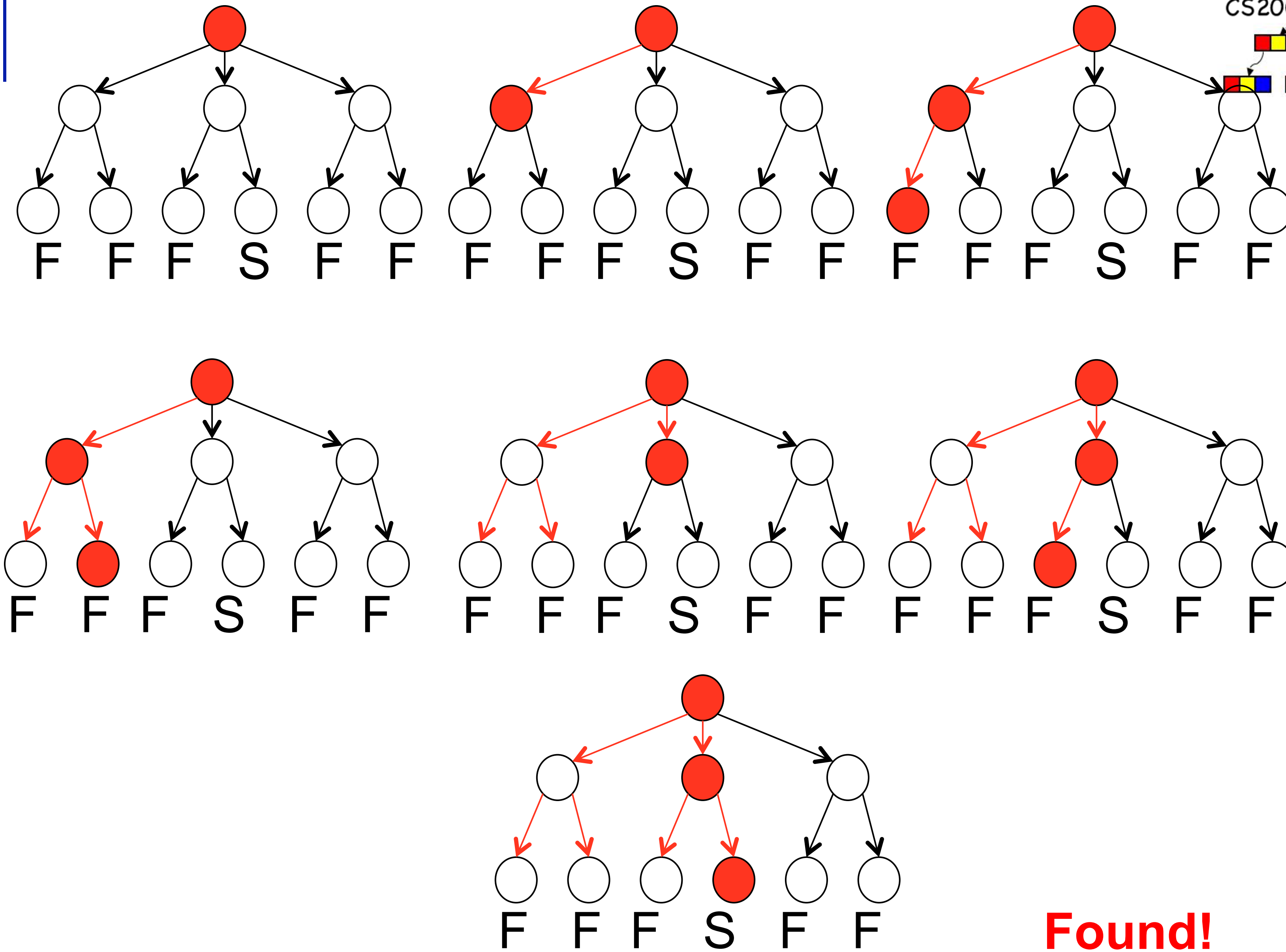
Think of it as walking through a tree shaped state space.



3 guesses here

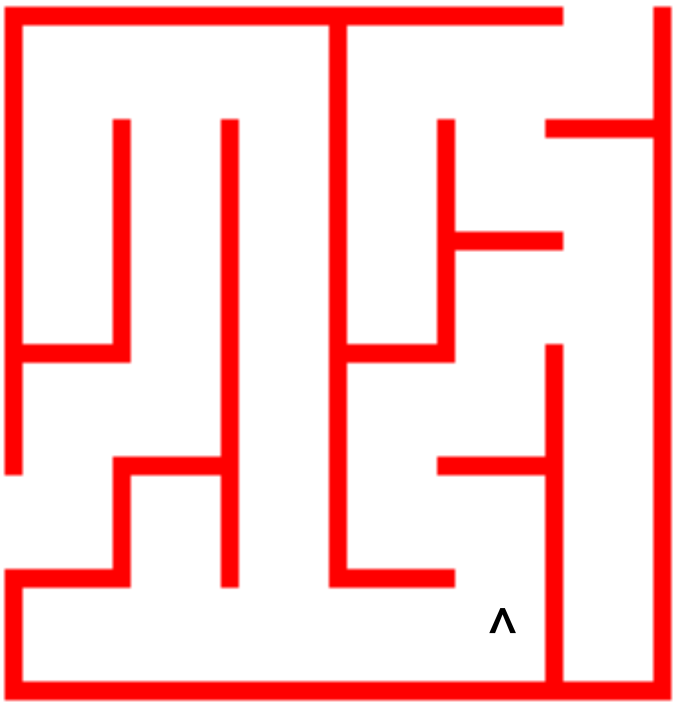
2 guesses in each state here

leaf states can fail (F) or succeed (S)


F F F S F F
F F F S F F
F F F S F F
F F F S F F
F F F S F F
F F F S F F
F F F S F F
Found!

Depth First Search

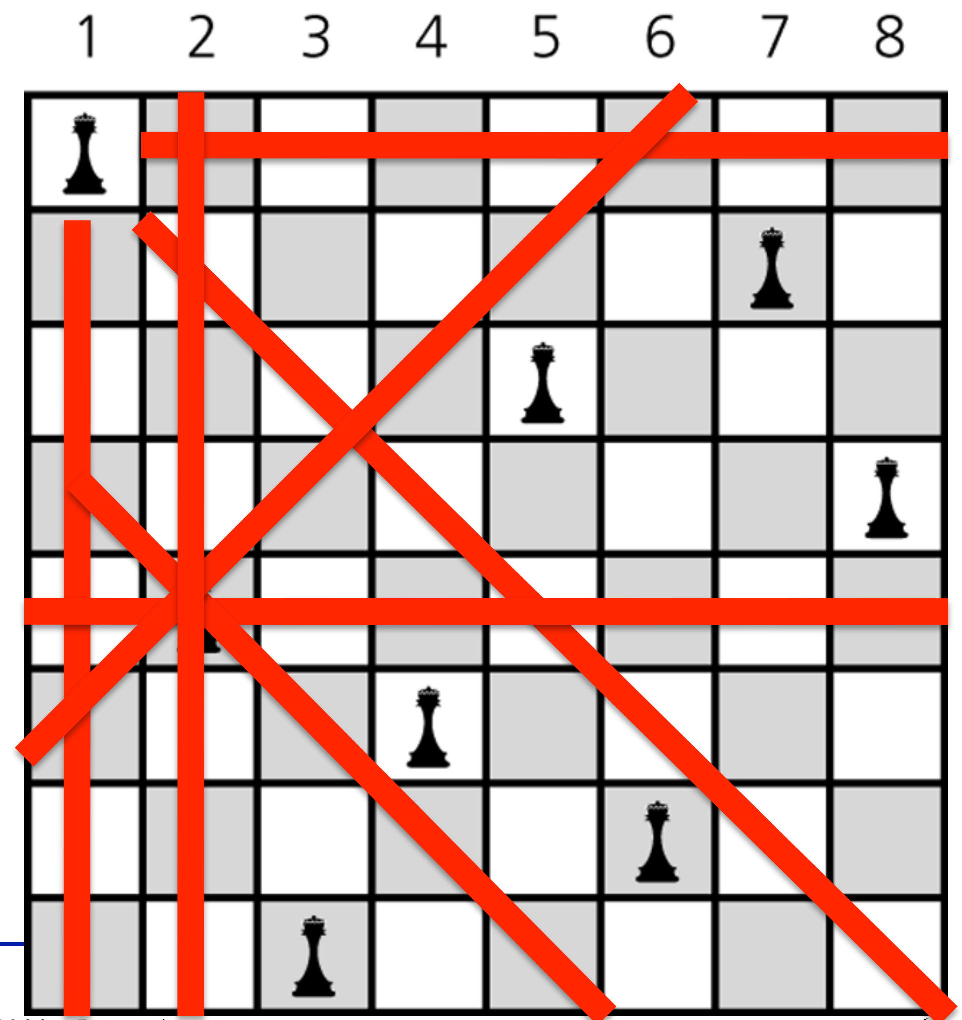
- Looking for a path out of the maze
- Strategy:
  - Prioritize directions: right, straight or left.
  - At a dead end "backtrack" to last not fully explored state and try a different direction
- Recursive solution?



The Eight Queens Problem

Place 8 Queens!

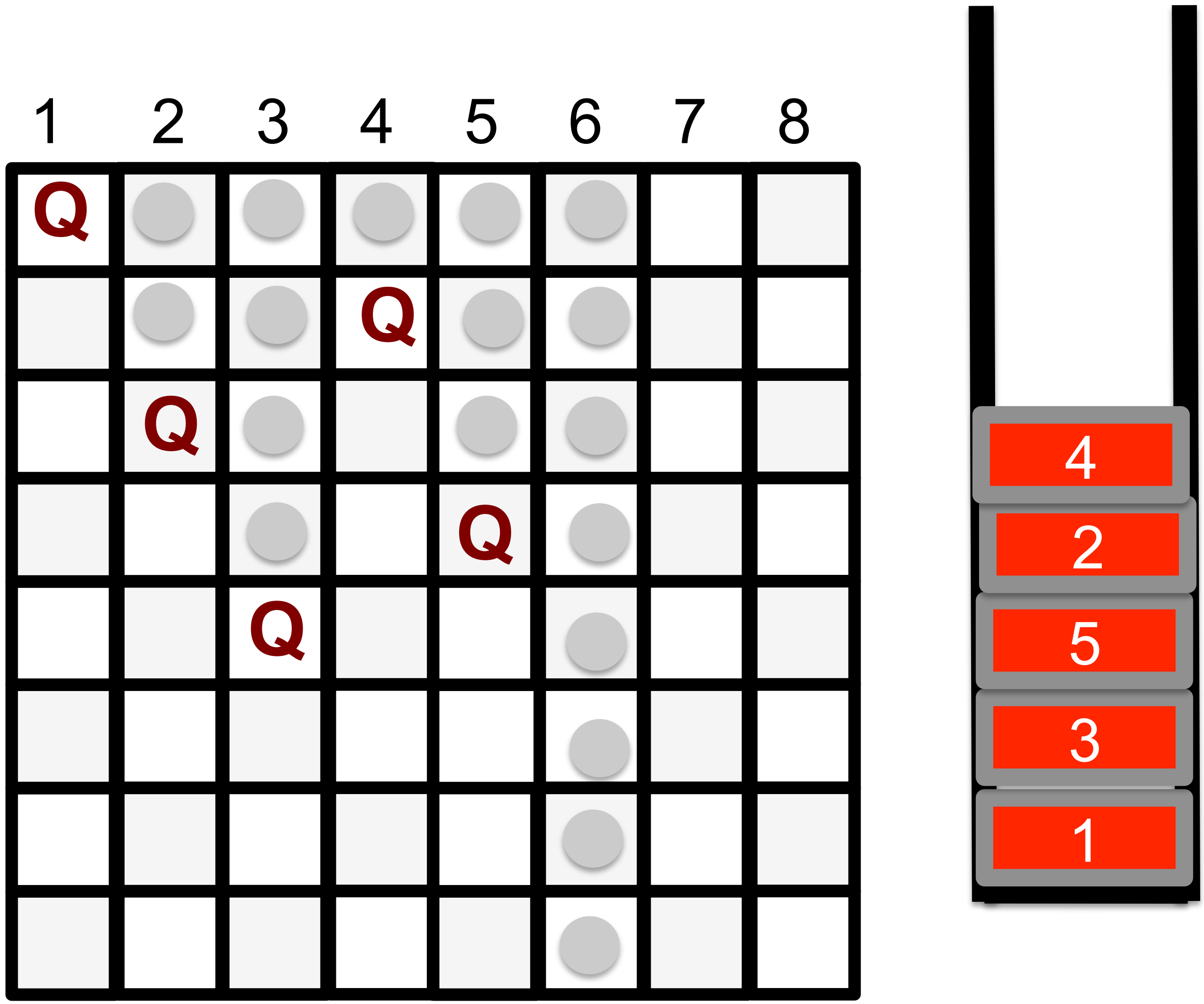
No queen can attack any other queens.



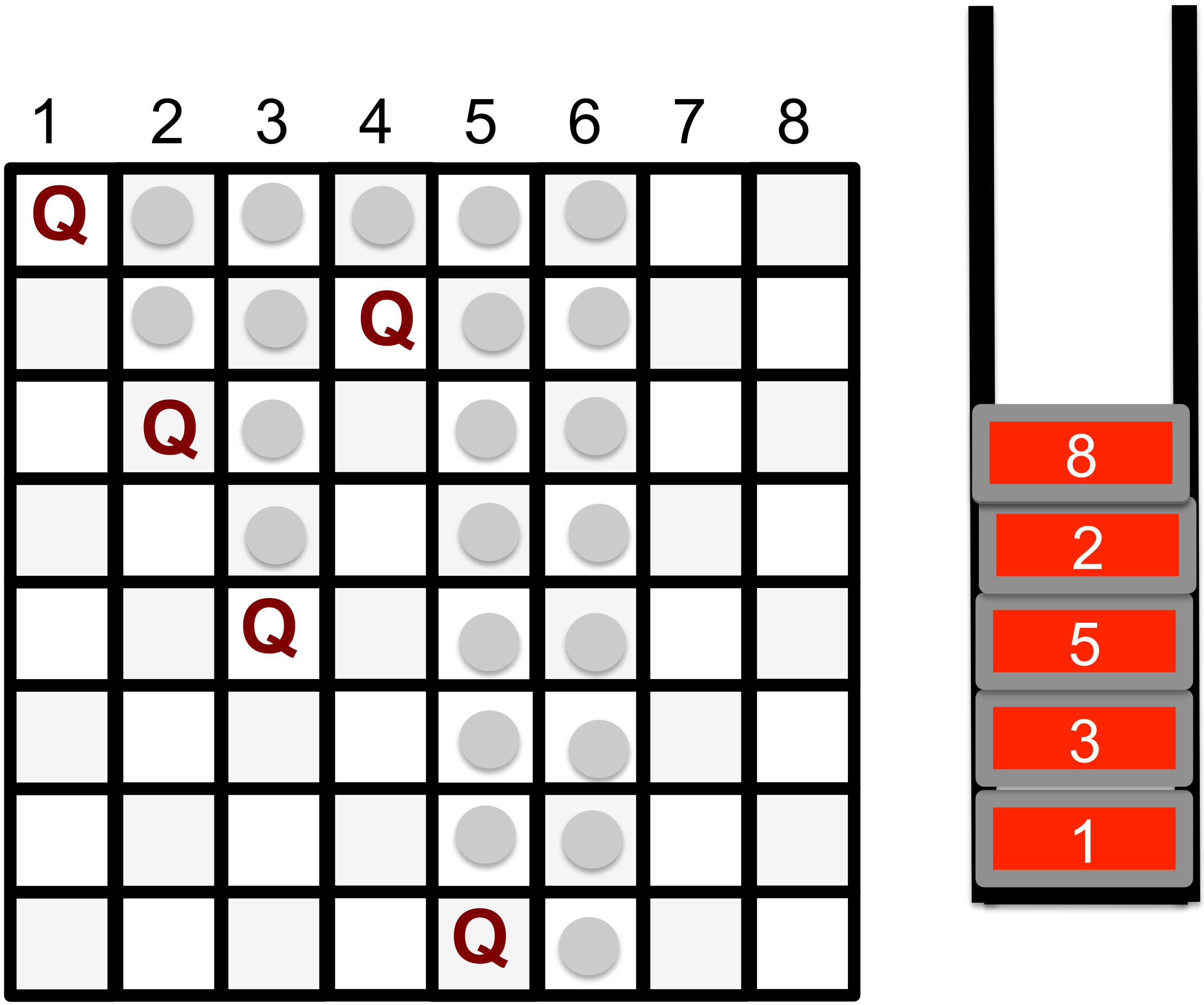
Solution with recursion and backtracking

```
placeQueen (in currColumn:integer)
if ( currColumn > 8) {
   The problem is solved
} else {
    while (unconsidered squares exist in currColumn) {
        Determine if the next square is safe.
        if (such a square exists){
           place a queen in the square
           placeQueens(currColumn+1) // try next column
           if (no queen safe in currColumn+1) {
                  remove queen from currColumn
                 try the next square in that column
           }
        }
     }
}
```

Example

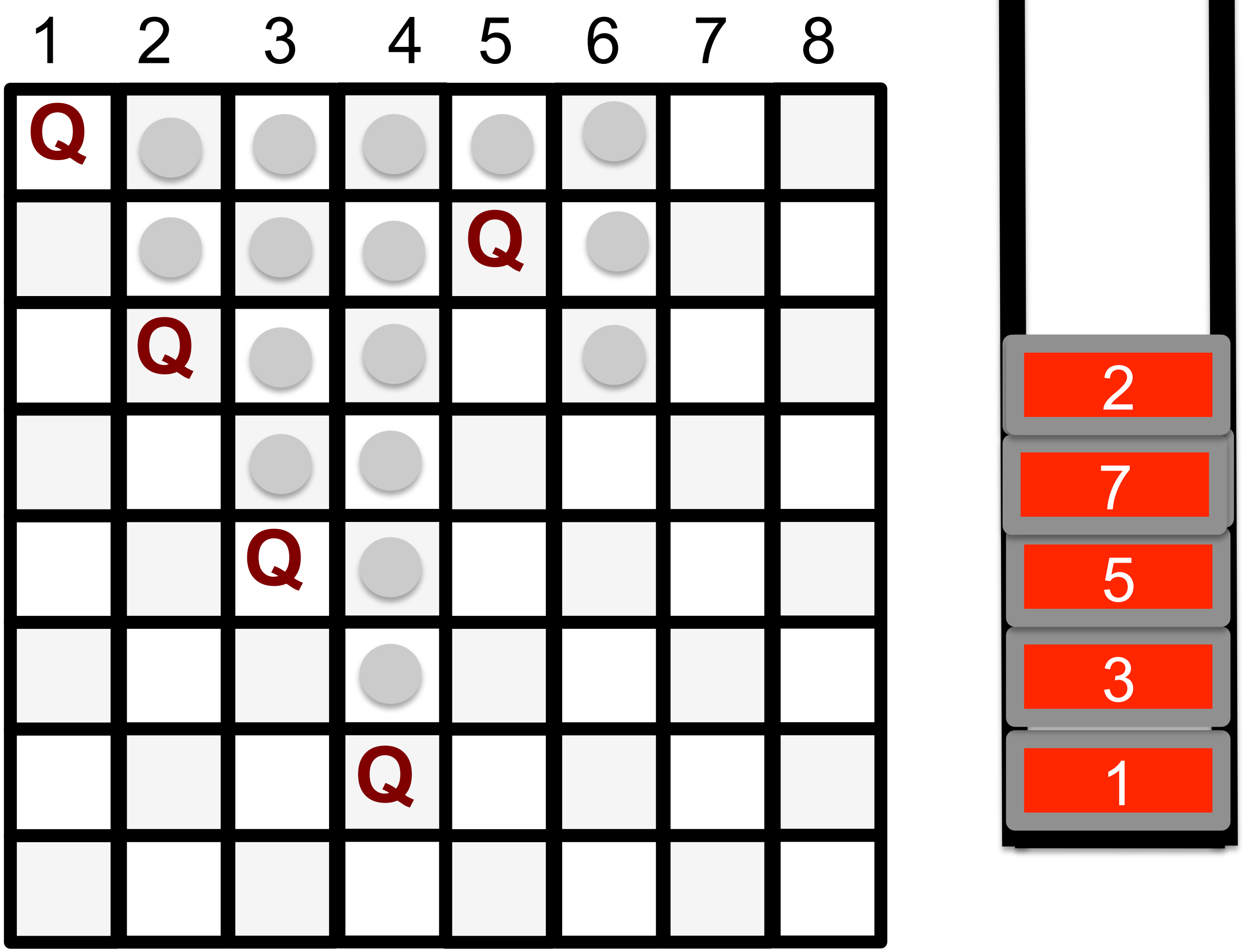


Hit ‘Dead End’



Backtrack

If you go on, this will fail and you need to back track to col 4



Backtrack: an 8 queens solution

					Q		
			Q				
						Q	
Q							
							Q
	Q						
				Q			
		Q					

The only symmetric one

There are 11 more "fundamental" solutions

see:

wikipedia.org/wiki/Eight_queens_puzzle

Questions

- What is the maximum depth of the run time stack for 8 Queens?
- How big could the call tree get?

Recursion

- Specifies a solution to one or more base cases
- Then demonstrates how to derive the solution to a problem of an arbitrary size
  - From solutions to smaller sized problems.

Correctness of the Recursive Factorial Method

Specification of the problem
(e.g., Mathematical definition, SW requirements)

Algorithm
(e.g., pseudo code)

Does your algorithm satisfy the specification of the problem?

Correctness of the Recursive Factorial Method

Definition of Factorial

$factorial(n) = n\ (n - 1)\ (n - 2)\ \ldots\ 1$ for any integer $n > 0$

$factorial(0) = 1$

Definition of method $fact(N)$

```
int fact(int n){
      if (n == 0)
         return 1;
      else
         return n* fact(n-1);
}
```

Inductive proof fact computes the factorial of its argument.

Basis step:

$fact(0) = 1$

Inductive Step:

Show that for an arbitrary positive integer k,

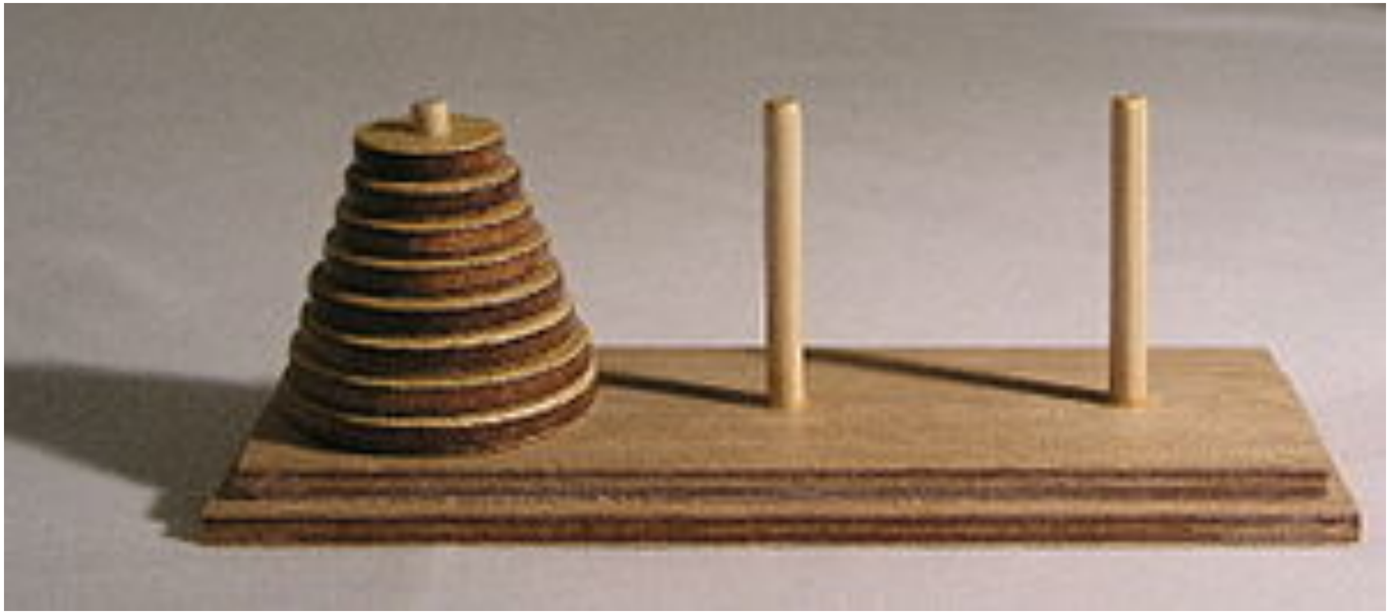
if $fact(k)$ returns $k!$, then

$fact(k+1)$ returns $(k+1)!$

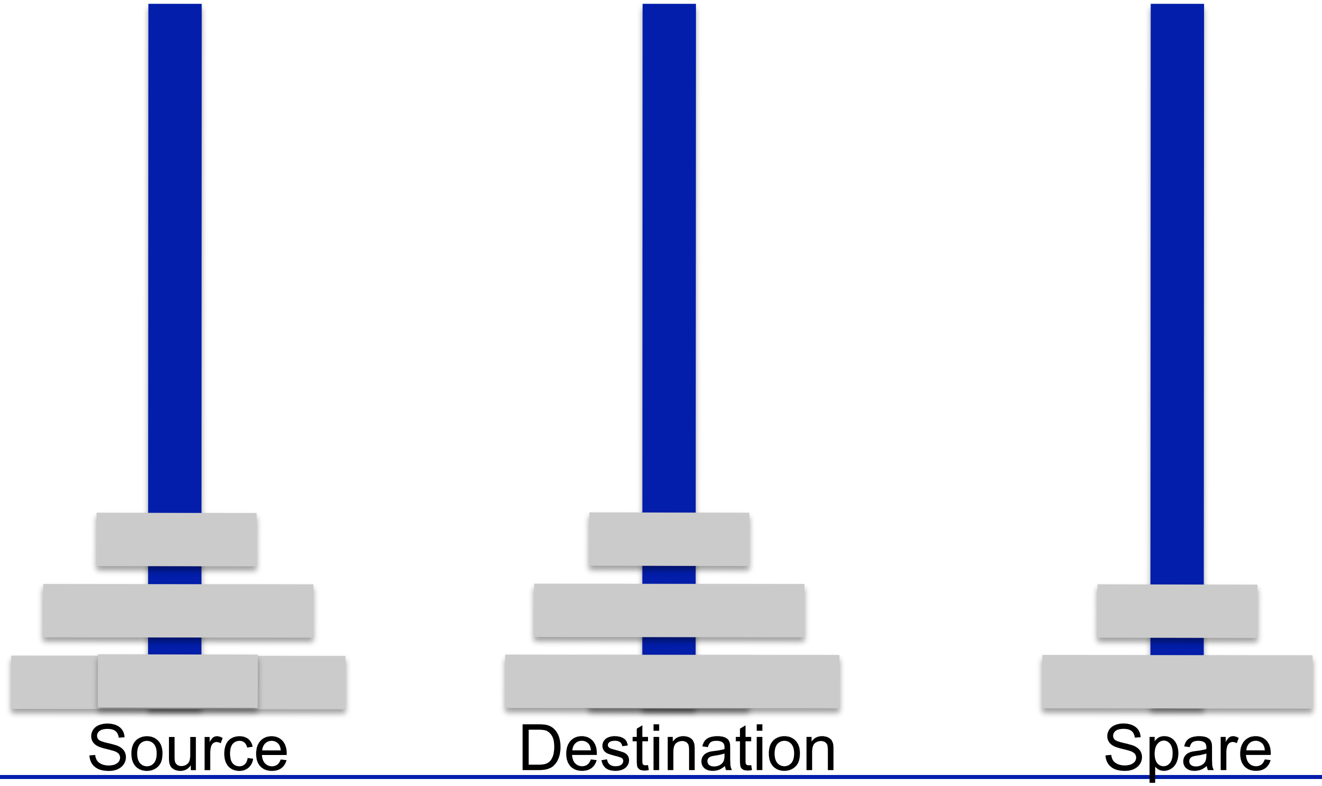
do it do it

The Towers of Hanoi Example

- **Move pile of disks from source to destination**
- **Only one** disk may be moved at a time.
- No disk may be placed on top of a smaller disk.



States in the Towers of Hanoi



Recursive Solution

```
// pegs are numbers, via is computed
// number of moves are counted
// empty base case
public void hanoi(int n, int from, int to){
  if (n>0) {
    int via = 6 - from - to;
    hanoi(n-1,from, via);
    System.out.println("move disk " + n + " from " + from + " to " + to);
    hanoi(n-1,via,to);
  }
}
```

let’s run it and

study the move pattern, and count the number of moves

Cost of Towers of Hanoi

- How many moves does *hanoi(n)* make?
- *from the recursive code:*

  $moves(1) = 1$

  $moves(N) = moves(N\text{-}1)+1+moves(N\text{-}1)$ *(if* $N>1$*)*

- By inspection, we can infer that a closed form formula for the number of moves:

  $moves(N) = 2^N - 1$ *(for all* $N>=1$*)*

- Can we prove it?

Proof

- Basis Step
  - Show that the property is true for N = 1.

    $2^1 - 1 = 1$, which is consistent with the recurrence relation's specification that $moves(1) = 1$

- Inductive Step
  - Property is true for an arbitrary k ➔ property is true for $k+1$
  - Assume that the property is true for N = k

    $moves(k) = 2^k - 1$
  - Show that the property is true for $N = k + 1$
  - *Do it, do it*

Proof – cont.

- $moves(k+1) = 2 * moves(k) + 1$

$$= 2 * (2^k - 1) + 1$$

$$= 2 * 2^k - 2 + 1 = \mathbf{2^{k+1} - 1}$$

Therefore the inductive proof is complete.

One more example:

$$0+1+2\ldots+n = n(n+1)/2 \qquad n=0,1,2\ldots.$$

base: $0 = 0*1/2=0$ Check

step: assume: $0+1+2\ldots+k = k(k+1)/2$

show that $0+1+2\ldots+k+ (k+1) = (k+1)(k+2)/2$

$0+1+2\ldots+k+ (k+1) = k(k+1)/2 + (k+1) =$

$k(k+1)/2 + 2(k+1)/2 = k(k+1)/2 + 2(k+1)/2 =$

$(k+2)(k+1)/2 = (k+1)(k+2)/2$

Check