CS 220: Discrete Structures and their Applications

Predicate Logic
Section 1.6-1.10 in zybooks
From propositional to predicate logic

Let’s consider the statement

“x is an odd number”

Its truth value depends on the value of the variable x. Once we assign x a value, it becomes a proposition.

Predicate logic will allow us to reason about statements with variables without having to assign values to them.
Predicates

**Predicate:** A logical statement whose truth value is a function of one or more variables.

**Examples:**
- \(x\) is an odd number
- Computer \(x\) is under attack
- The distance between cities \(x\) and \(y\) is less than \(z\)

When the variables are assigned a value, the predicate becomes a proposition and can be assigned a truth value.
The truth value of a predicate can be expressed as a function of the variables, for example:

“x is an odd number”
can be expressed as $P(x)$.

So, the statement $P(5)$ is the same as "5 is an odd number".

“The distance between cities x and y is less than z miles”
Represented by a predicate function $D(x, y, z)$

$D($fort-collins, denver, 100$)$ is true.
The domain of a variable in a predicate is the set of all possible values for the variable.

Examples:
The domain of the predicate "x is an odd number" is the set of all integers.
In general, the domain of a predicate should be defined with the predicate.

Compare domain to type. Compare predicate to a java method declaration.

What about the predicate:
The distance between cities x and y is less than z miles?
Consider the predicate $S(x, y, z)$ which is the statement that “$x + y = z$”

What is the domain of the variables in the predicate?

What is the truth value of:

- $S(1, -1, 0)$
- $S(1, 2, 5)$
Let $R(x, y)$ denote: $x$ beats $y$ in rock-paper-scissors with the standard rules.

- What are the truth values of:
  - $R(\text{Rock, Paper})$
  - $R(\text{Scissors, Paper})$

Uses of predicate logic

Verifying program correctness

- Consider the following snippet of code:
  ```java
  if (x < 0)
    x = -x;
  ```

- What is true before? (called **precondition**)
  - x has some value

- What is true after? (called **postcondition**)
  - greaterThan(x, 0)
Assigning values to variables is one way to provide them with a truth value.

Alternative: Say that a predicate is satisfied for every value (universal quantification), or that it holds for some value (existential quantification)

Example:
Let $P(x)$ be the statement $x + 1 > x$. This holds regardless of the value of $x$. We express this as: $\forall x \; P(x)$
Universal quantification is the statement “P(x) for all values of x in the domain of P”

Notation: $\forall x \ P(x)$
$\forall$ is called the universal quantifier

If the domain of P contains a finite number of elements $a_1, a_2, \ldots, a_k$:
$\forall x \ P(x) \equiv P(a_1) \land P(a_2) \land \ldots \land P(a_k)$
Universal quantification is the statement "P(x) for all values of x in the domain of P"

Notation: $\forall x \ P(x)$

$\forall$ is called the universal quantifier

An element $x$ for which $P(x)$ is false is called a *counterexample*.

Example: Let $P$ be the predicate "$x^2 > x$" with the domain of real numbers. Give a counterexample.

What does the existence of a counterexample tell us about the truth value of $\forall x \ P(x)$?
Existential quantification of $P(x)$ is the statement
There exists an element $x$ in the domain of $P$ such that $P(x)$

Notation: $\exists x\ P(x)$
$\exists$ is called the existential quantifier

Example:
$M(x)$ - “$x$ is a mammal” and
$E(x)$ - “$x$ lays eggs”
(both with the domain of “animals”).
What is the truth value of $\exists x\ (M(x) \land E(x))$?

True (Platipus)
Existential quantification

Existential quantification of $P(x)$ is the statement

There exists an element $x$ in the domain of $P$ such that $P(x)$

Notation: $\exists x \ P(x)$
$\exists$ is called the existential quantifier

If the domain of $P$ contains a finite number of elements $a_1, a_2, ..., a_k$:

$\exists x \ P(x) \equiv P(a_1) \lor P(a_2) \lor ..., \lor P(a_k)$
Consider the following predicates:

$P(x)$: $x$ is prime

$O(x)$: $x$ is odd

The proposition $\exists x \ (P(x) \land \neg O(x))$ states that there exists a positive number that is prime and not odd.

Is this true?

What about $\forall x \ (P(x) \rightarrow O(x))$?
The quantifiers : $\exists$ and $\forall$ have higher precedence than the logical operators from propositional logic.

Therefore:

$\forall x \ P(x) \land Q(x)$ means:

$(\forall x \ P(x)) \land Q(x)$ rather than:

$\forall x \ (P(x) \land Q(x))$
When a quantifier is used on a variable x, we say that this occurrence of x is **bound**

All variables that occur in a predicate must be bound or assigned a value to turn it into a proposition

Example: $\exists x \ D(x, \ Denver, \ 60)$
Examples

In the statement $\exists x \ (x + y = 1)$ $x$ is bound

In the statement $\exists x \ P(x) \lor \forall x \ R(x)$ all variables are bound
Can also be written as: $\exists x \ P(x) \lor \forall y \ R(y)$

What about $\forall x \ P(x) \land Q(x)$?

Better to express this as
What about $\forall x \ P(x) \land Q(y)$?
English to Logic

Every student in CS220 has visited Mexico
Every student in CS220 has visited Mexico or Canada
Suppose we want to negate the statement:

“Every student in CS220 has taken Math160”

Translation into logic:

\( \forall x \ P(x) \) where \( P \) is the predicate “\( x \) has taken Math160”, with the domain of CS220 students.

The negation: “not every student in CS220 has taken Math160”, or “there exists a student in CS220 who hasn’t taken Math160” i.e.:

\( \exists x \ \neg P(x) \)
Note

Alternative way of expressing the statement
“Every student in CS220 has taken Math160”

\[ \forall x \ (\text{takes}(x, \text{CS220}) \rightarrow \text{hasTaken}(x, \text{math160})) \]

or

\[ \forall x \ (\text{takesCS220}(x) \rightarrow \text{hasTakenMath160}(x)) \]
De Morgan’s laws for quantifiers

We have illustrated the logical equivalence:

$$\neg \forall x \ P(x) \equiv \exists x \ \neg P(x)$$

A similar equivalence holds for the existential quantifier:

$$\neg \exists x \ P(x) \equiv \forall x \ \neg P(x).$$

Example: There does not exist someone who likes to go to the dentist. Same as: everyone does not like to go to the dentist.
De Morgan’s laws for quantifiers

Example:
Each quantifier be expressed using the other

\( \forall x \text{ Likes}(x, \text{IceCream}) \quad \neg \exists x \neg \text{Likes}(x, \text{IceCream}) \)

\( \exists x \text{ Likes}(x, \text{Broccoli}) \quad \neg \forall x \neg \text{Likes}(x, \text{Broccoli}) \)
No one in this class is wearing shorts and a ski parka.
Some lions do not drink coffee
Nested quantifiers

If a predicate has more than one variable, each variable must be bound by a separate quantifier.

\[ \forall x \exists y P(x,y) \quad x \text{ and } y \text{ are both bound.} \]
\[ \forall x P(x,y) \quad x \text{ is bound and } y \text{ is free.} \]
\[ \exists y \exists z T(x,y,z) \quad y \text{ and } z \text{ are bound. } x \text{ is free.} \]

The logical expression is a proposition if all the variables are bound.
Nested quantifiers of the same type

Example:

$M(x, y)$: $x$ sent an email to $y$, with the domain of people.

Consider the statement

$$\forall x \forall y M(x, y)$$

In English: Every person sent an email to everyone.

This is a statement on all pairs $x, y$:

For every pair of people, $x, y$ it is true that $x$ sent $y$ a mail.
Nested quantifiers

Example:

$M(x, y)$: $x$ sent an email to $y$, with the domain of people.

Consider the statement

$$\forall x \forall y \ M(x, y)$$

In English: Every person sent an email to everyone - including themselves. But what if we would like to exclude the self emails?

$$\forall x \forall y \ ((x \neq y) \rightarrow M(x, y))$$
Nested quantifiers of the same type

Example:
\( M(x, y) \): \( x \) sent an email to \( y \), with the domain of people.

Express the following in English:
\( \exists x \ \exists y \ M(x, y) \)

Order does not matter:
\( \forall x \ \forall y \) is the same as \( \forall y \ \forall x \)
\( \exists x \ \exists y \) is the same as \( \exists y \ \exists x \)
Alternating nested quantifiers

$\exists x \ \forall y \text{ is not the same as } \forall y \ \exists x$:  

$\exists x \ \forall y \ \text{Likes}(x, y)  
  \begin{itemize} 
    \item “There is a person who likes everyone”
  \end{itemize}$  

$\forall y \ \exists x \ \text{Likes}(x, y)  
  \begin{itemize} 
    \item “Everyone is liked by at least one person”
  \end{itemize}$
Nested quantifiers as a two person game

Two players: the Universal player, and the Existential player. Each selects values for their variables.

Example: $\forall x \, \exists y \, (x + y = 0)$

The Universal player selects a value. Given the value chosen, the Existential player chooses its value, making the statement true.
Nested quantifiers as a two person game

Two players: the Universal player, and the Existential player. Each selects values for their variables.

What happens in this situation: $\exists x \, \forall y \, (x + y = 0)$?
Expressing uniqueness

$L(x)$: $x$ was late

How do we express the statement that exactly one person was late?

What’s wrong with $\exists x \ L(x)$?
Expressing uniqueness

$L(x)$: $x$ was late

How do we express the statement that exactly one person was late?

What's wrong with $\exists x \ L(x)$?

Instead:

$\exists x \ (L(x) \land \forall y((x \neq y) \rightarrow \neg L(y)))$
Moving quantifiers

$L(x): x$ was late

How do we express the statement that exactly one person was late?

$\exists x \ (L(x) \land \forall y ((x \neq y) \rightarrow \neg L(y)))$

Equivalent to:

$\exists x \ \forall y \ (L(x) \land ((x \neq y) \rightarrow \neg L(y)))$
De Morgan’s laws with nested quantifiers

\[ \neg \forall x \ \forall y \ P(x, y) \equiv \exists x \ \exists y \ \neg P(x, y) \]
\[ \neg \forall x \ \exists y \ P(x, y) \equiv \exists x \ \forall y \ \neg P(x, y) \]
\[ \neg \exists x \ \forall y \ P(x, y) \equiv \forall x \ \exists y \ \neg P(x, y) \]
\[ \neg \exists x \ \exists y \ P(x, y) \equiv \forall x \ \forall y \ \neg P(x, y) \]

Example:
\[ \exists x \ \forall y \ \text{Likes}(x, y) \]: There is a person who likes everyone.

Its negation:
\[ \neg \exists x \ \forall y \ \text{Likes}(x, y) \]: There is no person who likes everyone.
\[ \neg \exists x \ \forall y \ \text{Likes}(x, y) \equiv \forall x \ \exists y \ \neg \text{Likes}(x, y) \]: Every person has someone that they do not like.