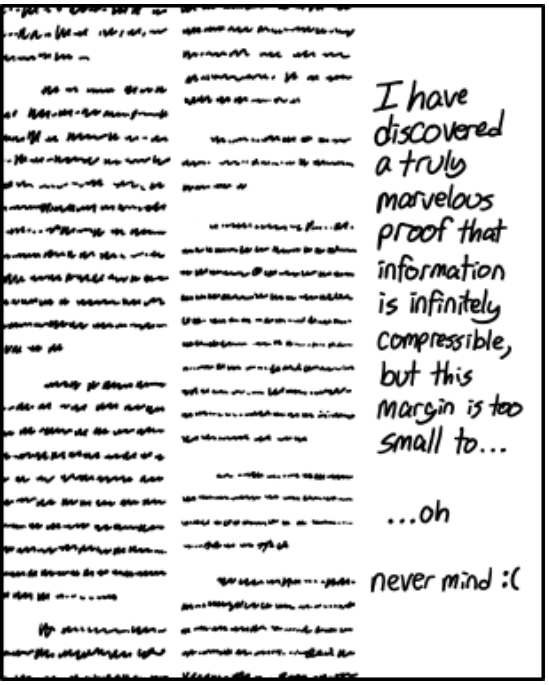
CS 220: Discrete Structures and their Applications

Proofs

sections 2.4 – 2.5 in zybooks



https://xkcd.com/1381/

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Terminology

Theorem: statement that can be shown to be true

Proof: a valid argument that establishes the truth of a theorem

Conjecture: statement *believed* to be a true

Proof techniques

Proof techniques:

- Direct proof
- Proof by contrapositive
- Proof by contradiction
- Proof by cases
- Proof by induction – later in the course

Proof by contradiction

Prove: "If $n=ab$ where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$."

- Proof by contradiction.
- Suppose the conclusion is false.
- i.e. the statement "$a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false.
- Hence, using DeMorgan's law, $a > \sqrt{n}$ and $b > \sqrt{n}$
- Hence $ab > n$, which contradicts $n=ab$

Proof by contradiction

The process:

- ✓ Assume that the theorem is false.
- ✓ Show that some logical inconsistency arises as a result.

If t is the statement of the theorem, the proof begins with the assumption $\neg t$ and leads to a conclusion that $\neg r \wedge r$, for some proposition r.

Proof by contradiction

The process:

- ✓ Assume that the theorem is false.
- ✓ Show that some logical inconsistency arises as a result.

If t is the statement of the theorem, the proof begins with the assumption $\neg t$ and leads to a conclusion that $\neg r \wedge r$, for some proposition r.

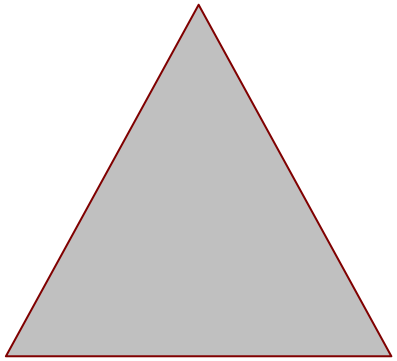
In other words, we get that $\neg t \rightarrow (\neg r \wedge r)$

The only way for this to be true is for $\neg t$ to be false, showing that t is true.

Proof by contrapositive can be seen as a special case.

Example

Theorem: Every triangle has at least one acute (less than 90 degrees) angle.



Example

Theorem: Among any group of 25 people, there must be at least three who are all born in the same month.

(Pigeon hole principle)

Example

Theorem: There is no smallest positive real number.

$\sqrt{2}$ is irrational

Theorem: $\sqrt{2}$ is irrational.

Proof.

- Assume that "$\sqrt{2}$ is irrational" is false, that is, $\sqrt{2}$ is rational.
- Hence, $\sqrt{2} = a/b$ such that a and b have no common factors (if they have then we can divide these out)
- So $2b^2 = a^2$, which means a^2 must be even
- Lemma: if n^2 is even then n is even
- Hence, $a = 2c$
- Therefore $b^2 = 2c^2$ so b^2 must be even.
- Therefore, according to the lemma, b is even, which means a and b have common factor 2, which is a contradiction.

Proof by cases

Prove: For every integer x, x^2-x is an even integer.

Idea:

Let's reason about the parity of x^2-x for two cases:

Case 1: x is even.

Case 2: x is odd

Proof by cases

Prove: For every integer x, x^2-x is an even integer.

Let's reason about the parity of x^2-x for two cases:

Case 1: x is even, i.e. $x = 2k$ for some integer k

$x^2 - x = (2k)^2 - 2k = 4k^2 - 2k = 2(2k^2 - k)$

Proof by cases

Prove: For every integer x, x^2-x is an even integer.

Let's reason about the parity of x^2-x for two cases:

Case 2: x is odd, i.e. $x = 2k+1$ for some integer k

$$x^2 - x = (2k+1)^2 - (2k + 1) = 4k^2 + 4k + 1 - 2k - 1 = 2(2k^2 - k)$$

Example

Theorem: Consider a group of six people. Each pair of people are either friends or enemies with each other. Then there are three people in the group who are all mutual friends or all mutual enemies.

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

What are the cases?

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

Case 1: $x \geq 0, y \geq 0$

Case 2: $x < 0, y \geq 0$

Case 3: $x \geq 0, y < 0$

Case 4: $x < 0, y < 0$

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

Case 1: $x \geq 0$, $y \geq 0$

In this case $x*y \geq 0$; so $|x*y|=x*y$

$|x|=x$ and $|y|=y$; so $|x|*|y|=x*y$.

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

Case 2: $x<0$, $y\geq 0$

In this case $|x*y| = -x*y$

$|x|=-x$ and $|y|=y$; so $|x|*|y|=-x*y$.

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

Case 1: $x \geq 0, y \geq 0$ ✓

Case 2: $x < 0, y \geq 0$ ✓

Case 3: $x \geq 0, y < 0$ ✓

This is analogous to case 2 with the roles of x and y reversed.

Proof by cases

Theorem: Given two real numbers x and y,

$$|x*y| = |x|*|y|$$

Case 4: $x<0$, $y<0$

In this case $x*y>0$; so $|x*y|=x*y$

$|x|=-x$ and $|y|=-y$; so $|x|*|y|=x*y$.

Example

Prove: For integers x and y, if xy is odd, then x is odd and y is odd.

Which proof technique would be best?