CS 220: Discrete Structures and their Applications

Sets
zybooks sections 4.1-4.7
Set: An unordered collection of objects
The objects in a set are called its members or elements.

Example: \{2, 4, 8\} is the set containing the elements 2, 4, 8
This form of specifying a set is called roster notation

\{2, 4, 8\} is the same set as \{4, 2, 8\} (unordered)
Set: An unordered collection of objects
The objects in a set are called its members or elements.

Notation for set membership $\in$

$a \in A$ means "a is an element of the set A."

$A = \{1, 2, 3, 4, 5\}$

$4 \in A$
Examples

V={a, e, i, o, u}  Set of vowels
B={False, True}  Boolean values
O={1, 3, 5, ..., 99}  Odd numbers between 1 and 99
Examples

The natural numbers \( N = \{0,1,2,3,\ldots\} \)

The integers \( Z = \{\ldots,-2,-1,0,1,2,\ldots\} \)

The positive integers \( Z^+ = \{1,2,\ldots\} \)

The rational numbers \( \mathbb{Q} \)

Cardinality of a set: number of distinct elements in the set. Denoted by \(|S|\).

A set is finite if its cardinality is finite (and infinite otherwise)

\[
A = \{ x \in N : x \leq 2000 \} \quad \text{what is } |A| \ ? \\
B = \{ x \in N : x \geq 2000 \} \quad \text{what is } |B| \ ?
\]
Building sets

Sometimes it’s hard to list all the elements of the set explicitly. E.g. the set of all odd numbers less than 100:

\[ O = \{1, 3, 5, \ldots, 99\} \]

Ellipsis “…” is used instead of the omitted elements.

Instead we can characterize the set by the property its elements satisfy:

\[ O = \{x : x \text{ is an odd positive integer less than 100}\} \]

This is called set builder notation.
Set builder notation

We can express the set \( O = \{1,3,5,\ldots,99\} \) using set builder notation:

\[
O = \{x \in \mathbb{Z}^+ : x \text{ is odd and } x < 100\}
\]

Definition of set builder notation:

\[
A = \{x \in S : P(x)\}
\]

or

\[
A = \{x \in S \mid P(x)\}
\]

\( S \) – a set

\( P(x) \) – a predicate

Example:

\[
D = \{ x \in \mathbb{R} : |x| < 1 \}
\]

Can also be written as:

\[
D = \{ x : x \in \mathbb{R} \text{ and } |x| < 1 \}
\]
A set $A$ is said to be a subset of a set $B$ if and only if every element of $A$ is also an element of $B$.

Notation: $A \subseteq B$

Using logic:

$$A \subseteq B \iff \forall x \ (x \in A \Rightarrow x \in B)$$

Example: $\{1, 2, 4\} \subseteq \{1, 2, 3, 4, 5\}$
Questions

\{1, 2, 3\} \subseteq \{2, 3\} ?
\{1, 2, 3\} \subseteq \{1, 2, 3\} ?

What can we say about the relationship between the cardinalities of A and B if A \subseteq B?
Proper subsets

A is a proper subset of B if \( A \subseteq B \) and there is an element of B that is not an element of A.

Notation: \( A \subset B \)

Example:
\[
\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}
\]
Venn diagrams

Graphical representation of sets
$U$ - the set of all objects

$A \subseteq B$
Example

integers between 1 and 9

1 2 3 5 7

2 even

4 6 8

odd

9 prime
Set equality

Two sets are equal if and only if they have the same elements.

We write $A = B$ to denote set equality.

Using logic:

$$A = B \iff \forall x \ (x \in A \iff x \in B)$$
The empty set has no elements.
Notation: \( \{\} \) or \( \emptyset \)
Is \( \emptyset \subseteq \{1, 2, 3\} \)? Yes! Since
\[ \forall x \ x \in \emptyset \Rightarrow x \in \{1, 2, 3\} \]
In fact, this is true for any set.

The cardinality of \( \emptyset \) is zero: \( |\emptyset| = 0 \).
Questions

Is \{a\} \subseteq \{a\}?
Is \{a\} \in \{a\}?
Is \{a\} \in \{a,\{a\}\}?
sets of sets

As the previous example suggests, a set can have a set as an element!

Example:

\[ A = \{ \{ 1, 2 \}, \emptyset, \{ 1, 2, 3 \}, \{ 1 \} \} \]

Note the following:

\[ 1 \notin A \]
\[ \{ 1 \} \in A \]
\[ \{ 1 \} \not\subseteq A \text{ since } 1 \notin A. \]

The empty set \( \emptyset \) is not the same as \( \{ \emptyset \} \)
The power set of a set S is the set of all subsets of S.

Notation: \( P(S) \)

Examples:

\[
P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}
\]
\[
P(\emptyset) = \{\emptyset\}
\]

Theorem: Let \( A \) be a set of cardinality \( n \), then \( |P(A)| = 2^n \).
The power set

Video game example:

- Given there are four objects a player could pick up, what are all the possible states the player could be in with respect to the set of objects $O = \{\text{coin, apple, sword, shield}\}$
- Answer: $P(O)$
Set Operations
set intersection

The intersection of sets $A$ and $B$ is the set containing those elements that are in both $A$ and $B$.

Notation: $A \cap B$

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$.

Example: $\{1,2,3\} \cap \{1,3,5\} = \{1, 3\}$

Two sets are called disjoint if their intersection is the empty set.
You can take the intersection of infinite sets:

\[ A = \{ x \in \mathbb{Z} : x \text{ is a multiple of 2} \} \]

\[ B = \{ x \in \mathbb{Z} : x \text{ is a multiple of 3} \} \]

\[ A \cap B = \{ x \in \mathbb{Z} : x \text{ is a multiple of 6} \} \]
The union of sets A and B is the set that contains those elements that are either in A or in B, or in both.

- Notation: $A \cup B$
- $A \cup B = \{ x : x \in A \text{ or } x \in B \}.$

Example: $\{1,2,3\} \cup \{1,3,5\} = \{1,2,3,5\}$
operations on multiple sets

The use of parentheses is important!

E.g., what is $A \cap B \cup C$?
intersection/union of many sets

applying the intersection/union operations to large numbers of sets:

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{ x : x \in A_i \text{ for all } 1 \leq i \leq n \} \]

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \{ x : x \in A_i \text{ for some } 1 \leq i \leq n \} \]
The **difference** of sets $A$ and $B$ is the set containing those elements that are in $A$ but not in $B$.

$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$.

**Example:** \{a, b, c, e, f\} - \{d, e, f, g\} = \{a, b, c\}
symmetric difference

The difference operation is not commutative since it is not necessarily the case that \( A - B = B - A \).

Check this in the diagram

The **symmetric difference** between two sets, \( A \) and \( B \), denoted \( A \oplus B \), is the set of elements that are a member of exactly one of \( A \) and \( B \), but not both.

Also defined as:

\[
A \oplus B = ( A - B ) \cup ( B - A )
\]

Check it again
set complement

The universal set: the set of all elements in some domain (e.g. positive integers)

The complement of a set $A$ is the set of all elements in the universal set $U$ that are not elements of $A$.

\[ \bar{A} \]

Notation:
An alternative definition: $U - A$
Example:

What is the complement of the natural numbers \((\mathbb{N})\) with respect to the integers \((\mathbb{Z})\)?
# Summary of Set Operations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersection</td>
<td>$A \cap B$</td>
<td>{ $x : x \in A \text{ and } x \in B$ }</td>
</tr>
<tr>
<td>Union</td>
<td>$A \cup B$</td>
<td>{ $x : x \in A \text{ or } x \in B \text{ or both}$ }</td>
</tr>
<tr>
<td>Difference</td>
<td>$A - B$</td>
<td>{ $x : x \in A \text{ and } x \notin B$ }</td>
</tr>
<tr>
<td>Symmetric difference</td>
<td>$A \oplus B$</td>
<td>{ $x : x \in A - B \text{ or } x \in B - A$ }</td>
</tr>
<tr>
<td>Complement</td>
<td>$\bar{A}$</td>
<td>{ $x : x \notin A$ }</td>
</tr>
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</table>
expressing sets operations using logic

\[ x \in A \cap B \leftrightarrow (x \in A) \land (x \in B) \]
\[ x \in A \cup B \leftrightarrow (x \in A) \lor (x \in B) \]
\[ x \in \bar{A} \leftrightarrow \neg (x \in A) \]

The sets \( U \) and \( \emptyset \) correspond to the constants true (T) and false (F):

\[ x \in \emptyset \leftrightarrow F \]
\[ x \in U \leftrightarrow T \]
DeMorgan's laws for sets

We can use the laws of propositional logic to derive corresponding set identities:

\[ x \in \overline{A \cap B} \iff \neg (x \in A \cap B) \]
\[ \iff \neg (x \in A \land x \in B) \]
\[ \iff \neg (x \in A) \lor \neg (x \in B) \]
\[ \iff x \in \overline{A} \lor x \in \overline{B} \]
\[ \iff x \in (\overline{A} \cup \overline{B}) \]

Result:

\[ \overline{A \cap B} = \overline{A} \cup \overline{B} \]
# Set Identities

<table>
<thead>
<tr>
<th>Name</th>
<th>Identities</th>
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<tbody>
<tr>
<td>Idempotent laws</td>
<td>$A \cup A = A$</td>
<td>$A \cap A = A$</td>
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<tr>
<td>Associative laws</td>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
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<td>Commutative laws</td>
<td>$A \cup B = B \cup A$</td>
<td>$A \cap B = B \cap A$</td>
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<tr>
<td>Distributive laws</td>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
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<td>Identity laws</td>
<td>$A \cup \emptyset = A$</td>
<td>$A \cap U = A$</td>
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<tr>
<td>Domination laws</td>
<td>$A \cap \emptyset = \emptyset$</td>
<td>$A \cup U = U$</td>
</tr>
<tr>
<td>Double Complement law</td>
<td>$\overline{\overline{A}} = A$</td>
<td></td>
</tr>
<tr>
<td>Complement laws</td>
<td>$A \cap \overline{A} = \emptyset$</td>
<td>$A \cup \overline{A} = U$</td>
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<tr>
<td></td>
<td>$\overline{U} = \emptyset$</td>
<td>$\overline{\emptyset} = U$</td>
</tr>
<tr>
<td>De Morgan's laws</td>
<td>$\overline{A \cup B} = \overline{A} \cap \overline{B}$</td>
<td>$\overline{A \cap B} = \overline{A} \cup \overline{B}$</td>
</tr>
<tr>
<td>Absorption laws</td>
<td>$A \cup (A \cap B) = A$</td>
<td>$A \cap (A \cup B) = A$</td>
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Every set identity has a corresponding rule of propositional logic

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| Idempotent laws       | \( p \lor p = p \)                             | \( p \land p = p \)                            |
| Associative laws      | \((p \lor q) \lor r = p \lor (q \lor r)\)      | \((p \land q) \land r = p \land (q \land r)\) |
| Commutative laws      | \( p \lor q = q \lor p \)                      | \( p \land q = q \land p \)                    |
| Distributive laws     | \( p \lor (q \land r) = (p \lor q) \land (p \lor r) \) | \( p \land (q \lor r) = (p \land q) \lor (p \land r) \) |
| Identity laws         | \( p \lor F = p \)                             | \( p \land T = p \)                            |
| Domination laws       | \( p \land F = F \)                            | \( p \lor T = T \)                             |
| Double negation law   | \( \neg \neg p = p \)                          |                                                 |
| Complement laws       | \( p \land \neg p = F \)                       | \( p \lor \neg p = T \)                        |
|                       | \( \neg T = F \)                               | \( \neg F = T \)                               |
| De Morgan's laws      | \( \neg (p \lor q) = \neg p \land \neg q \)   | \( \neg (p \land q) = \neg p \lor \neg q \)   |
| Absorption laws       | \( p \lor (p \land q) = p \)                   | \( p \land (p \lor q) = p \)                    |
If order matters:

An ordered $n$-tuple is a sequence of $n$ objects

$(a_1, a_2, ..., a_n)$

First component is $a_1$

...  

$n$-th component is $a_n$

An ordered pair: 2-tuple $(a, b)$
An ordered triple: 3-tuple $(a, b, c)$

Sets do not have the same element more than once:

$\{1, 1, 2\} = \{1, 2\}$

Tuples can have the same element more than once:

$(1, 1, 1)$ is a valid 3-tuple
Two tuples are equal iff corresponding pairs of elements are equal:

\[(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \text{ iff } a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n\]

\[(2, 1) \neq (1, 2), \text{ but } \{2, 1\} = \{1, 2\}\]

Think of tuples as book chapters and sections

\[(1, 1): \quad \text{Chapter 1, section 1}\]
\[(1, 2, 4): \quad \text{Chapter 1, section 2, sub-section 4}\]
The **cartesian product** of sets $A$ and $B$ is denoted by $A \times B$ and is defined as:

$$\{ (a, b) : a \in A \text{ and } b \in B \}$$

**Example:** $A = \{1, 2\}$, $B = \{a, b, c\}$

$$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$$

Is $A \times B$ the same as $B \times A$?
Cartesian product of the sets
A = \{x, y, z\} and
B = \{1, 2, 3\}

![Cartesian product image from Wikipedia](https://en.wikipedia.org/wiki/Cartesian_product)
The cartesian product $\mathbb{R} \times \mathbb{R}$ (\(\mathbb{R}\) being the real numbers) gives every point in a 2D plane a pair of \(x, y\) coordinates:
Fact: $|A \times B| = |A| \times |B|$

Example: $A = \{1, 2\}$, $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
The cartesian product of sets $A_1, \ldots, A_n$ is the set of $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1, 2, \ldots, n$.

Denoted by $A_1 \times A_2 \times \ldots \times A_n$

Example: $A = \{0, 1\}$, $B = \{2, 3\}$, $C = \{4, 5, 6\}$

What is $A \times B \times C$?

What is $|A \times B \times C|$?
You can take the cartesian product of a set with itself.

Given a set $A$ we can look at $A \times A$ (denoted $A^2$), and more generally

$A \times A \times \ldots \times A$ denoted as $A^k$.

$k$ times

Example: if $A = \{0, 1\}$, then $A^k$ is the set of all ordered $k$-tuples whose entries are bits (0 or 1).

$\{0, 1\}^3 = \{ (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \}$

Example: $\mathbb{R}^2$ is the set of all points in the plane.
If $A$ is a set of symbols, then members of $A^k$ can be written without commas/parentheses.

For example:
If $A = \{0, 1\}$ then we can express $A^2$ as $\{00, 01, 10, 11\}$. 
Two sets, $A$ and $B$, are said to be disjoint if their intersection is empty ($A \cap B = \emptyset$).

A collection of sets, $A_1, A_2, ..., A_n$, is pairwise disjoint if every pair of sets is disjoint i.e., $A_i \cap A_j = \emptyset$ when $i \neq j$.

A partition of a non-empty set $A$ is a collection of non-empty subsets of $A$ such that each element of $A$ is in exactly one of the subsets.

$A_1, A_2, ..., A_n$ is a partition for a non-empty set $A$ if:

- $A_i \subseteq A$ for all $i$.
- $A_i \neq \emptyset$.
- $A_1, A_2, ..., A_n$ are pairwise disjoint.
- $A = A_1 \cup A_2 \cup ... \cup A_n$.
practice question:
Suppose that every student is assigned a unique 8-digit ID number.

\[ A_i : \text{the set students whose ID number begins with the digit } i. \]

Assume that for each digit, \( i \), there is at least one student whose ID starts with \( i \).

Do the sets \( A_0, \ldots, A_9 \) form a partition of the set of students?