CS 220: Discrete Structures and their Applications

Sets

zybooks sections 4.1-4.7

Colorado State University

sets

Set: An unordered collection of objects

The objects in a set are called its members or elements.

Example: $\{2, 4, 8\}$ is the set containing the elements 2, 4, 8

This form of specifying a set is called roster notation

$\{2, 4, 8\}$ is the same set as $\{4, 2, 8\}$ (unordered)

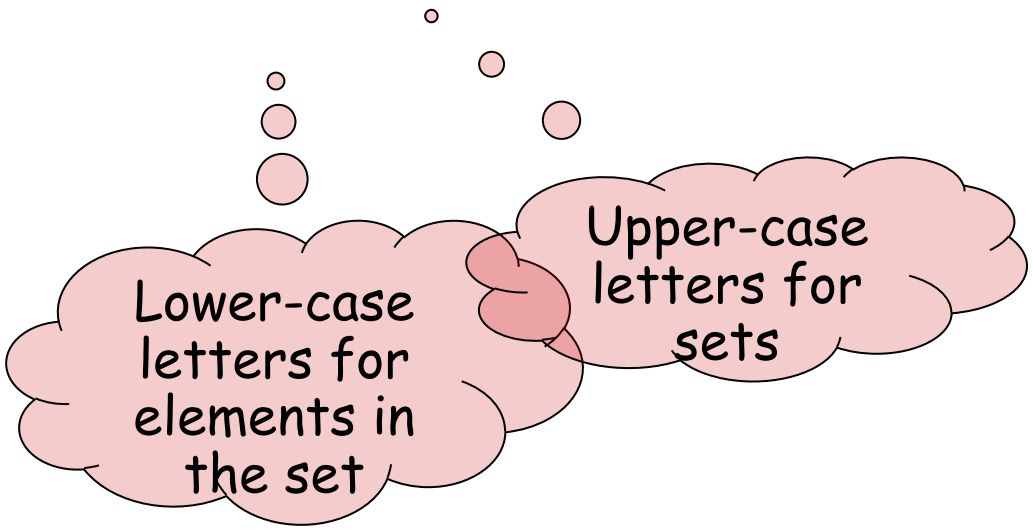
sets

Set: An unordered collection of objects

The objects in a set are called its members or elements.

Notation for set membership $\in$

$a \in A$ means "a is an element of the set A."



$A = \{1, 2, 3, 4, 5\}$

$4 \in A$

Examples

V={a, e, i, o, u}	Set of vowels
B={False, True}	Boolean values
O={1, 3, 5, ... ,99}	Odd numbers between 1 and 99

Examples

The natural numbers	$\mathbf{N} = \{0,1,2,3\ldots\}$	$\mathbb{N}$
The integers	$\mathbf{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$	$\mathbb{Z}$
The positive integers	$\mathbf{Z}^+ = \{1,2,\ldots\}$	
The rational numbers	$\mathbf{Q}$	$\mathbb{Q}$

Cardinality of a set: number of distinct elements in the set. Denoted by |S|.

A set is finite if its cardinality is finite (and infinite otherwise)

$A = \{ x \in \mathbf{N} : x \leq 2000 \}$ what is |A| ?

$B = \{ x \in \mathbf{N} : x \geq 2000 \}$ what is |B| ?

Building sets

Sometimes it's hard to list all the elements of the set explicitly

E.g. the set of all odd numbers less than 100:

$$O = \{1,3,5,\ldots,99\}$$

Ellipsis "…" is used instead of the omitted elements

Instead we can characterize the set by the property its elements satisfy:

$$O = \{x : x \text{ is an odd positive integer less than } 100\}$$

This is called set builder notation

Set builder notation

We can express the set O = {1,3,5,…,99} using set builder notation:

$$O = \{x \in \mathbf{Z}^+ : x \text{ is odd and } x < 100\}$$

Definition of set builder notation:

$$A = \{x \in S : P(x)\}$$

or

$$A = \{x \in S \mid P(x)\}$$

S – a set

P(x) – a predicate

Example: $D = \{ x \in R : |x| < 1 \}$

Can also be written as : $D = \{ x : x \in R \text{ and } |x| < 1 \}$

Subsets

A set A is said to be a subset of a set B if and only if every element of A is also an element of B.

Notation: $A \subseteq B$

Using logic:

$$A \subseteq B \leftrightarrow \forall x \ (x \in A \Rightarrow x \in B)$$

Example: $\{1, 2, 4\} \subseteq \{1, 2, 3, 4, 5\}$

Questions

$\{1, 2, 3\} \subseteq \{2, 3\}$?

$\{1, 2, 3\} \subseteq \{1, 2, 3\}$?

What can we say about the relationship between the cardinalities of A and B if $A \subseteq B$?

Proper subsets

A is a proper subset of B if $A \subseteq B$ and there is an element of B that is not an element of A.

Notation: $A \subset B$

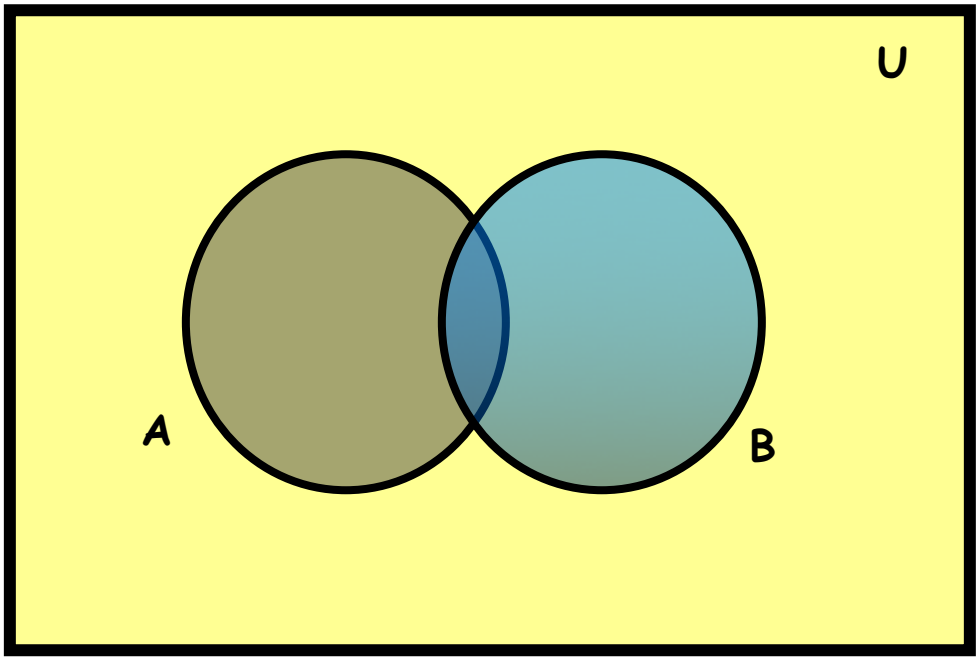
Example:

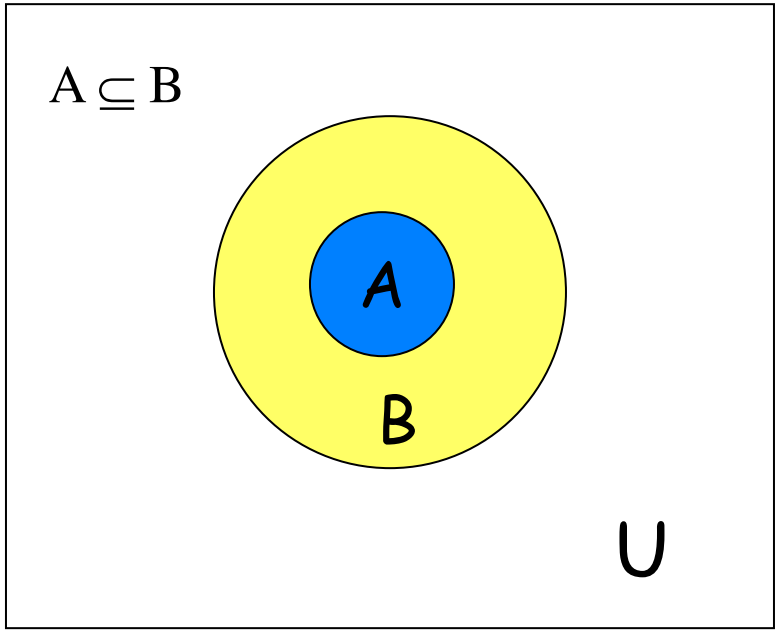
$$\{1, 2, 3\} \subset \{1, 2, 3, 4, 5\}$$

Venn diagrams

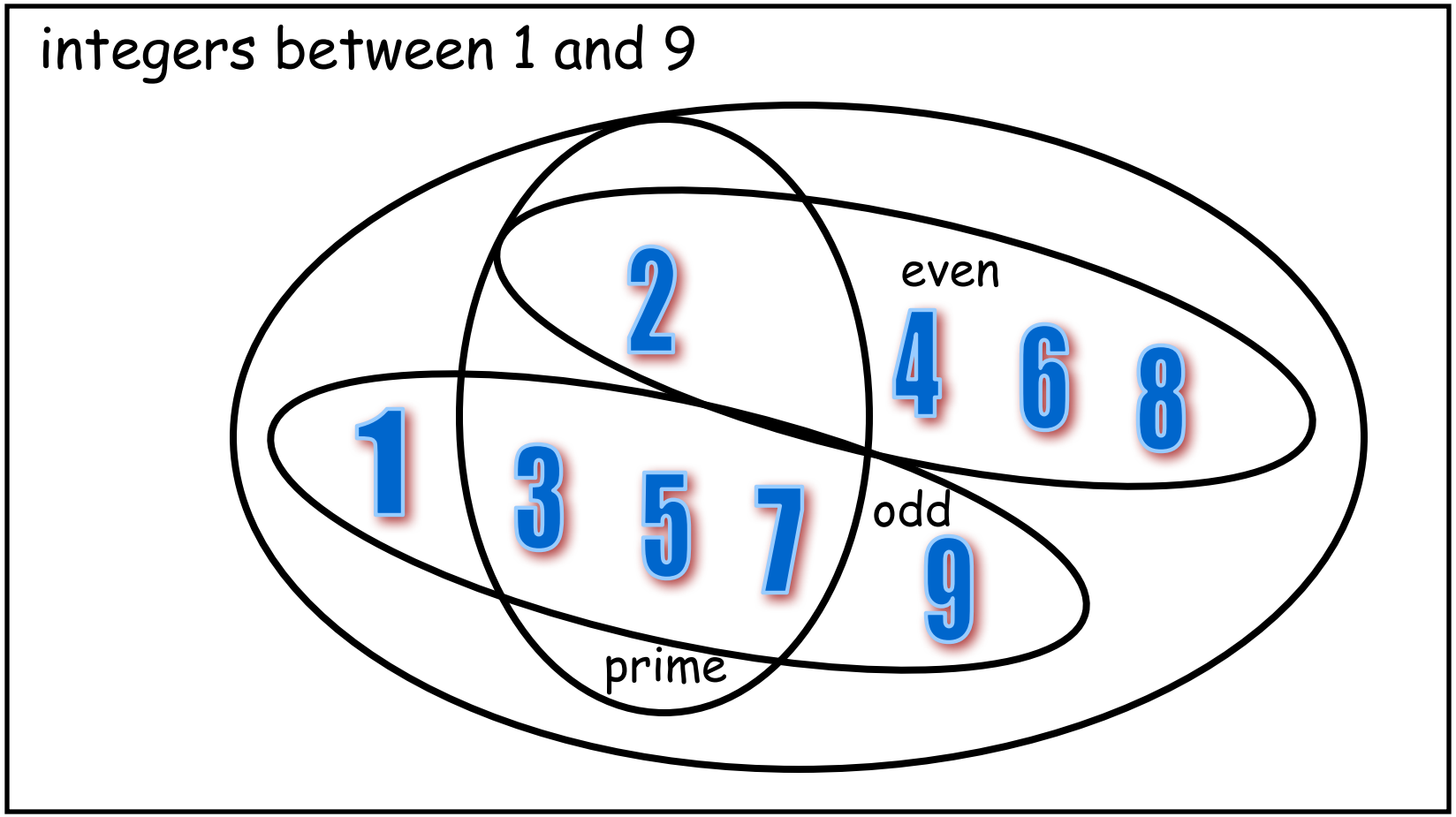
Graphical representation of sets

U – the set of all objects





Example



Set equality

Two sets are equal if and only if they have the same elements.

We write $A=B$ to denote set equality

Using logic:

$$A=B \leftrightarrow \forall x \ (x \in A \leftrightarrow x \in B)$$

The empty set

The empty set has no elements.

Notation: {} or $\varnothing$

Is $\varnothing \subseteq \{1,2,3\}$? Yes! Since

$$\forall x \;\; x \in \varnothing \Rightarrow \; x \in \{1, 2, 3\}$$

In fact, this is true for any set.

The cardinality of $\varnothing$ is zero: $|\varnothing| = 0$.

Questions

Is $\{a\} \subseteq \{a\}$?

Is $\{a\} \in \{a\}$?

Is $\{a\} \in \{a,\{a\}\}$?

sets of sets

As the previous example suggests, a set can have a set as an element!

Example:

$$A = \{ \{ 1, 2 \}, \varnothing , \{ 1, 2, 3 \}, \{ 1 \} \}$$

Note the following:

$1 \notin A$

$\{ 1 \} \in A$

$\{ 1 \} \not\subseteq A$ since $1 \notin A$.

The empty set $\varnothing$ is not the same as $\{ \varnothing \}$

The power set

The power set of a set S is the set of all subsets of S.

Notation: P(S)

Examples:

$$P(\{0,1,2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

$$P(\varnothing) = \{\varnothing\}$$

Theorem: Let A be a set of cardinality n, then $|P(A)| = 2^n$.

The power set

Video game example:

- Given there are four objects a player could pick up, what are all the possible states the player could be in with respect to the set of objects
  O = {coin, apple, sword, shield}
- Answer: P(O)

Set Operations

set intersection

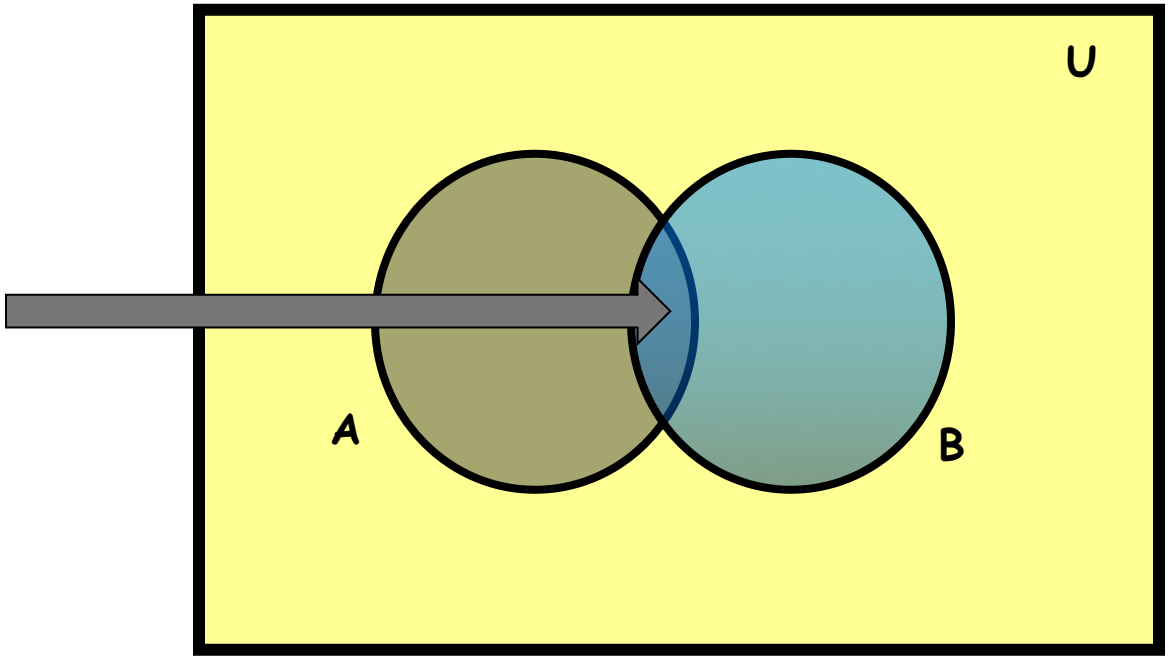
The intersection of sets A and B is the set containing those elements that are in both A and B.

Notation: $A \cap B$

$A \cap B = \{ x : x \in A \text{ and } x \in B\}$.

Example: $\{1,2,3\} \cap \{1,3,5\} = \{1, 3\}$

Two sets are called disjoint if their intersection is the empty set.



set intersection

You can take the intersection of infinite sets:

$A = \{ x \in \mathbf{Z} : x \text{ is a multiple of } 2 \}$

$B = \{ x \in \mathbf{Z} : x \text{ is a multiple of } 3 \}$

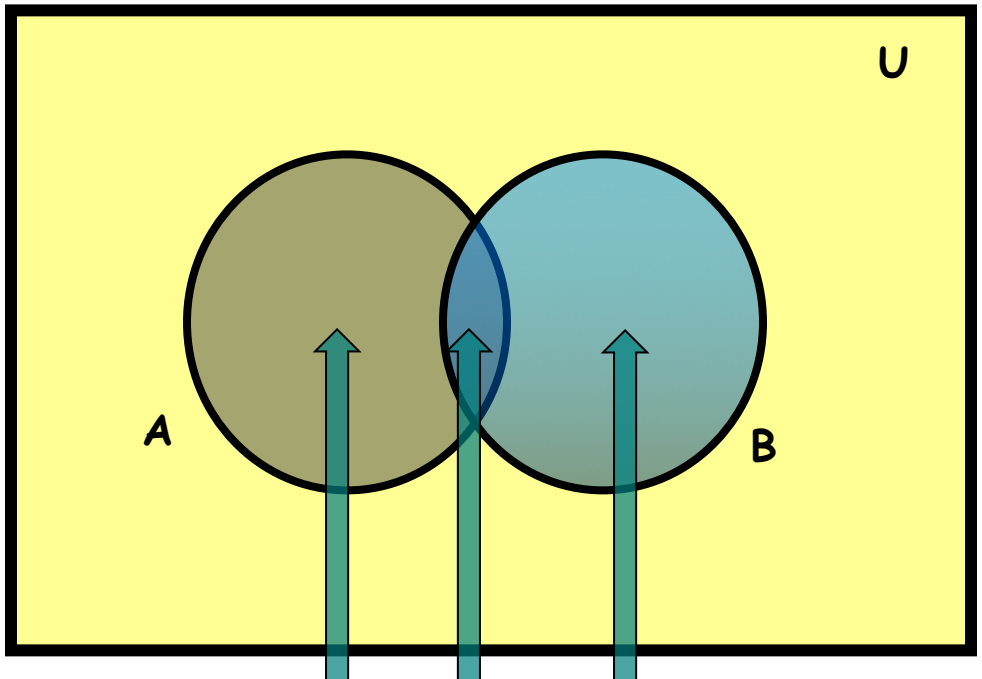
$A \cap B = \{ x \in \mathbf{Z} : x \text{ is a multiple of } 6 \}$

set union

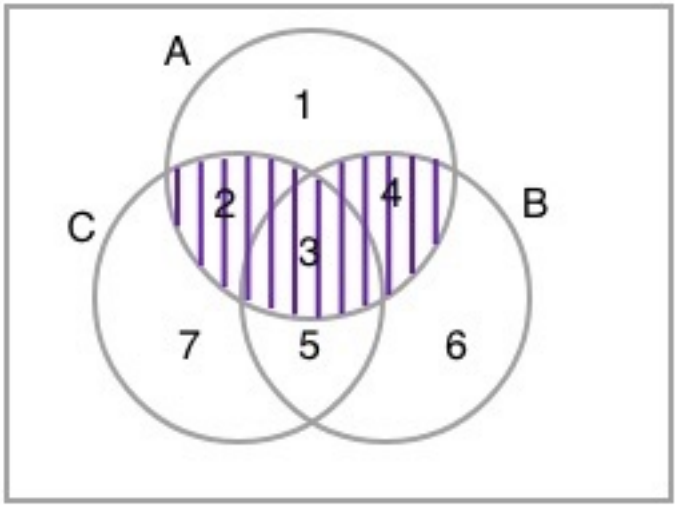
The union of sets **A** and **B** is the set that contains those elements that are either in **A** or in **B**, or in both.

- Notation: $A \cup B$
- $A \cup B = \{ x : x \in A \text{ or } x \in B \}$.

Example: $\{1,2,3\} \cup \{1,3,5\} = \{1,2,3,5\}$



operations on multiple sets



$A = \{ 1, 2, 3, 4 \}$

$B = \{ 3, 4, 5, 6 \}$

$C = \{ 2, 3, 5, 7 \}$

$A \cap (B \cup C)$

$B \cup C = \{2, 3, 4, 5, 6, 7\}$

$A = \{ 1, 2, 3, 4 \}$

$A \cap (B \cup C) = \{2, 3, 4\}$

The use of parentheses is important!

E.g., what is $A \cap B \cup C$?

intersection/union of many sets

applying the intersection/union operations to large numbers of sets:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{x : x \in A_i \text{ for all } 1 \leq i \leq n\}$$

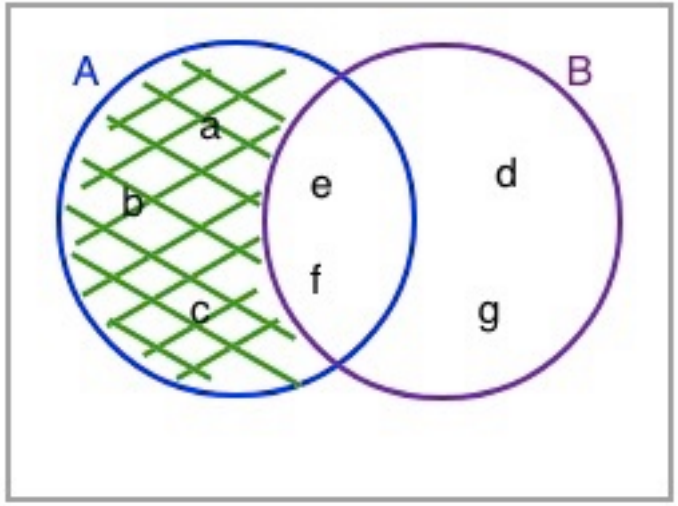
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \{x : x \in A_i \text{ for some } 1 \leq i \leq n\}$$

set difference

The difference of sets A and B is the set containing those elements that are in A but not in B.

$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$.

Example: $\{a, b, c, e, f\} - \{d, e, f, g\} = \{a, b, c\}$



symmetric difference

The difference operation is not commutative since it is not necessarily the case that $A - B = B - A$.

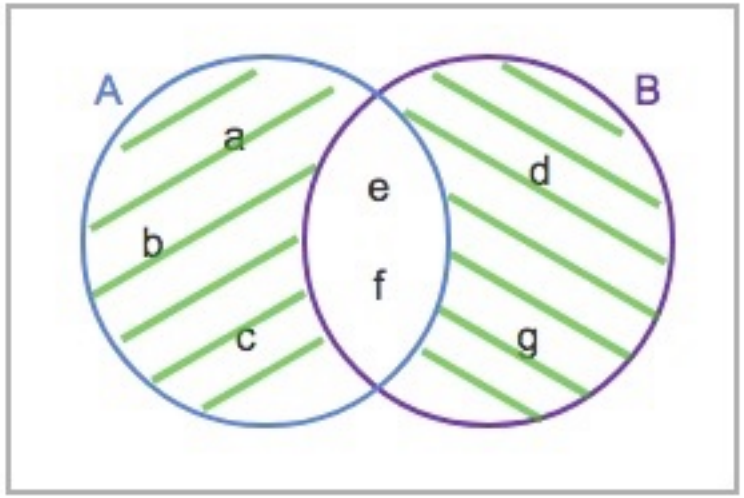
Check this in the diagram

The symmetric difference between two sets, A and B, denoted $A \oplus B$, is the set of elements that are a member of exactly one of A and B, but not both.

Also defined as:

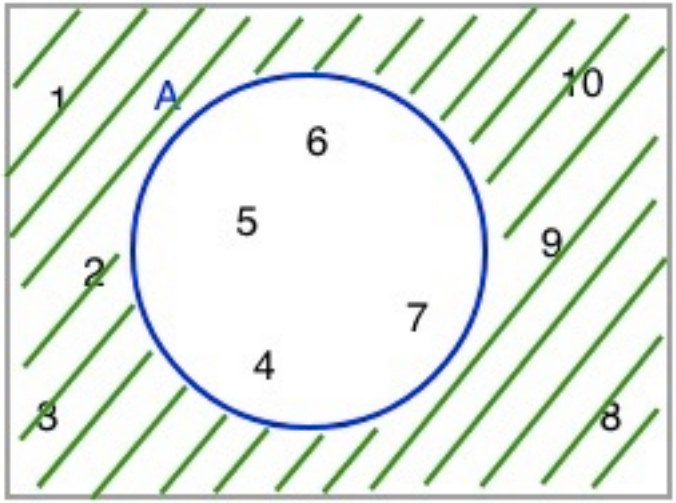
$A \oplus B = (A - B) \cup (B - A)$

Check it again



set complement

The universal set: the set of all elements in some domain (e.g. positive integers)



The complement of a set A is the set of all elements in the universal set U that are not elements of A.

Notation: $\bar{A}$

An alternative definition: U - A

set complement

Example:

What is the complement of the natural numbers (**N**) with respect to the integers (**Z**)?

summary of set operations

Operation	Notation	Description
Intersection	$A \cap B$	$\{x : x \in A \text{ and } x \in B\}$
Union	$A \cup B$	$\{x : x \in A \text{ or } x \in B \text{ or both}\}$
Difference	$A - B$	$\{x : x \in A \text{ and } x \notin B\}$
Symmetric difference	$A \oplus B$	$\{x : x \in A - B \text{ or } x \in B - A\}$
Complement	$\overline{A}$	$\{x : x \notin A\}$

expressing sets operations using logic

$$x \in A \cap B \leftrightarrow (x \in A) \wedge (x \in B)$$

$$x \in A \cup B \leftrightarrow (x \in A) \vee (x \in B)$$

$$x \in \bar{A} \leftrightarrow \neg(x \in A)$$

The sets U and $\varnothing$ correspond to the constants true (T) and false (F):

$$x \in \varnothing \leftrightarrow F$$

$$x \in U \leftrightarrow T$$

DeMorgan's laws for sets

We can use the laws of propositional logic to derive corresponding set identities:

$$
\begin{aligned}
x \in \overline{A \cap B} &\iff \neg(x \in A \cap B) \\
&\iff \neg(x \in A \wedge x \in B) \\
&\equiv \neg(x \in A) \vee \neg(x \in B) \\
&\iff x \in \overline{A} \vee x \in \overline{B} \\
&\iff x \in (\overline{A} \cup \overline{B})
\end{aligned}
$$

Result:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

set identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Domination laws	$A \cap \varnothing = \varnothing$	$A \cup U = U$
Double Complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \varnothing$ $\overline{U} = \varnothing$	$A \cup \overline{A} = U$ $\overline{\varnothing} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Domination laws	$A \cap \varnothing = \varnothing$	$A \cup U = U$
Double Complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \varnothing$ $\overline{U} = \varnothing$	$A \cup \overline{A} = U$ $\overline{\varnothing} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	

Every set identity has a corresponding rule of propositional logic

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$

tuples

If order matters:

An ordered n-tuple is a sequence of n objects

$$(a_1, a_2, \ldots, a_n)$$

First component is a_1

…

n-th component is a_n

An ordered pair: 2-tuple (a, b)

An ordered triple: 3-tuple (a, b, c)

Sets do not have the same element more than once:

$\{1, 1, 2\} = \{1,2\}$

Tuples can have the same element more than once:

(1,1,1) is a valid 3-tuple

tuples

Two tuples are equal iff corresponding pairs of elements are equal:

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff

$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$

$(2, 1) \neq (1, 2)$, but $\{2, 1\} = \{1, 2\}$

Think of tuples as book chapters and sections

(1, 1) : Chapter 1, section 1

(1, 2, 4): Chapter 1, section 2, sub-section 4

cartesian products

The cartesian product of sets A and B is denoted by A x B and is defined as:

$$\{ (a, b) : a \in A \text{ and } b \in B\}$$

Example: $A = \{1, 2\}$, $B = \{a, b, c\}$

$$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Is A x B the same as B x A?

cartesian products

Cartesian product of the sets

A = {x, y, z} and

B = {1, 2, 3}

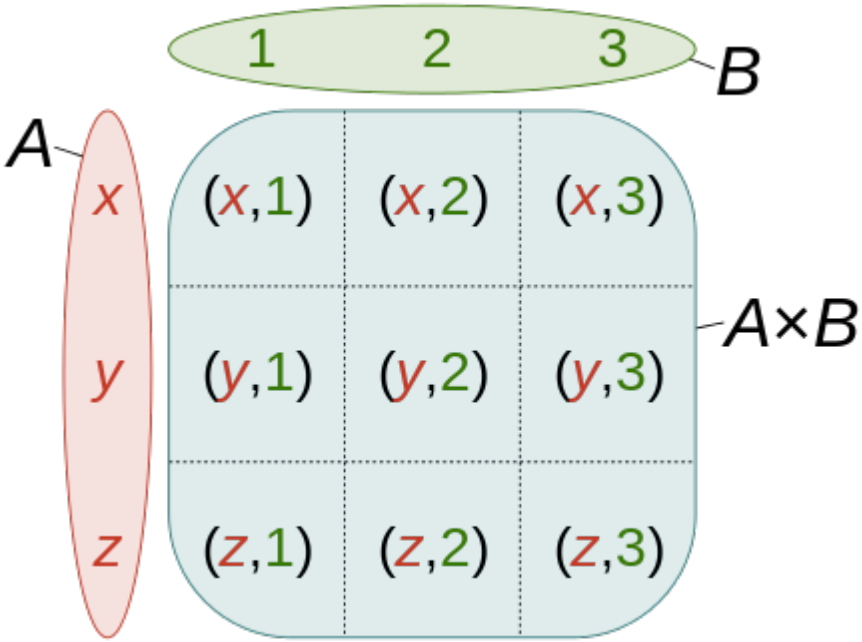
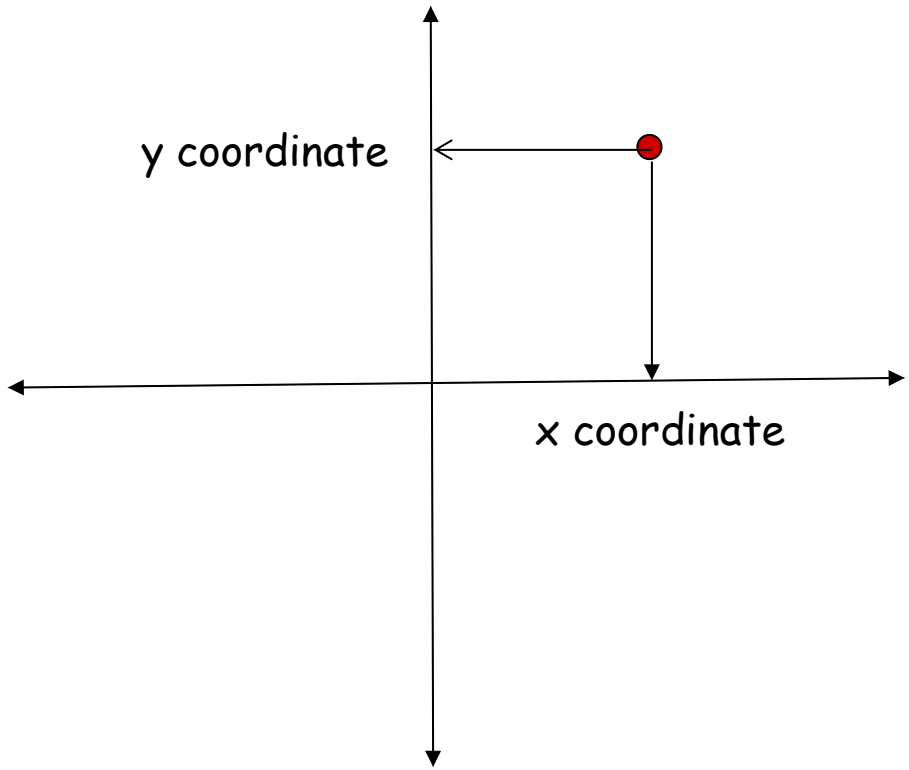


image from https://en.wikipedia.org/wiki/Cartesian_product

cartesian products

The cartesian product **R** x **R** (**R** being the real numbers) gives every point in a 2D plane a pair of x,y coordinates:



cartesian products

Fact: $|A \times B| = |A| * |B|$

Example: $A = \{1, 2\}$, $B = \{a, b, c\}$

$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

cartesian products

The cartesian product of sets $A_1,\ldots,A_n$ is the set of *n*-tuples $(a_1,a_2,\ldots,a_n)$, where $a_i \in A_i$ for i=1, 2, …, n.

Denoted by $A_1 \times A_2 \times,\ldots,\times A_n$

Example: $\mathbf{A}=\{0, 1\}$, $\mathbf{B}=\{2, 3\}$, $\mathbf{C}=\{4, 5, 6\}$

What is $A \times B \times C$?

What is $|A \times B \times C|$?

cartesian products of a set with itself

You can take the cartesian product of a set with itself.

Given a set A we can look at $A \times A$ (denoted A^2), and more generally

$\underbrace{A \times A \times \ldots \times A}_{k \text{ times}}$ denoted as A^k.

Example: if $A = \{0, 1\}$, then A^k is the set of all ordered k-tuples whose entries are bits (0 or 1).

$\{0, 1\}^3 = \{ (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \}$

Example: $\mathbf{R}^2$ is the set of all points in the plane.

strings

If A is a set of symbols, then members of A^k can be written without commas/parentheses.

For example:

If $A = \{0, 1\}$ then we can express A^2 as $\{00, 01, 10, 11\}$.

partitions

Two sets, A and B, are said to be disjoint if their intersection is empty ($A \cap B = \varnothing$).

A collection of sets, $A_1, A_2, \ldots, A_n$, is pairwise disjoint if every pair of sets is disjoint i.e., $A_i \cap A_j = \varnothing$ when $i \neq j$.

A partition of a non-empty set A is a collection of non-empty subsets of A such that each element of A is in exactly one of the subsets.

$A_1, A_2, \ldots, A_n$ is a partition for a non-empty set A if:

- ✓ $A_i \subseteq A$ for all i.
- ✓ $A_i \neq \varnothing$
- ✓ $A_1, A_2, \ldots, A_n$ are pairwise disjoint.
- ✓ $A = A_1 \cup A_2 \cup \ldots \cup A_n$

partitions

practice question:

Suppose that every student is assigned a unique 8-digit ID number.

A_i : the set students whose ID number begins with the digit i.

Assume that for each digit, i, there is at least one student whose ID starts with i.

Do the sets A_0, …, A_9 form a partition of the set of students?