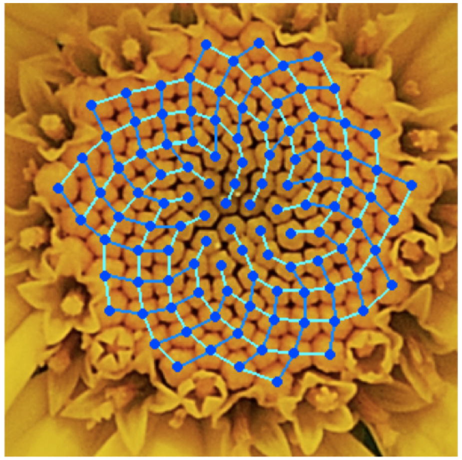
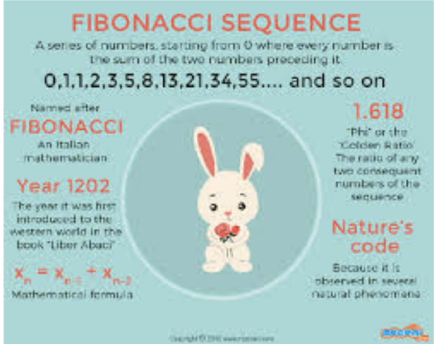
CS 220: Discrete Structures and their Applications

sequences, recurrence relations, summations

zybooks sections 6.1-6.3





sequences

A sequence is a special type of function in which the domain is a consecutive set of integers.

Example:

Consider a student's GPA in each of their four years in college.

Let's express this as a function $g : \{1,2,3,4\} \rightarrow \mathbf{R}$, e.g.

$$g(1)=3.67,\ \ g(2)=2.88,\ \ g(3)=3.25,\ \ g(4)=3.75$$

As a shorthand we'll use subscripts for the domain:

$$g_1=3.67,\ \ g_2=2.88,\ \ g_3=3.25,\ \ g_4=3.75$$

When the indices are known you can simply list the sequence of values:

$$3.67, 2.88, 3.25, 3.75$$

sequences

Sequences can have negative indices, e.g.

$a_{-2}=0,\ a_{-1}=1,\ a_0=1,\ a_1=0$

They can be finite:

$a_m,\ a_{m+1},\ \ldots\ ,a_n$

Or infinite:

$a_m,\ a_{m+1},\ a_{m+2},\ \ldots$

The elements of a sequence can be defined by a formula e.g.:

$d_k = 2^k$ where $k = 0,1,2,\ldots$

This defines the sequence 1, 2,4,....

geometric sequences

A geometric sequence is a sequence of real numbers of the form

$$a,\ ar, ar^2, \ldots, ar^n, \ldots$$

Each element is obtained by multiplying the previous element by the common ratio of the sequence (r); the first number is some arbitrary number (a)

Example:

1, 1/2, 1/4, 1/8, 1/16, ...

What are a and r for this sequence?

A geometric sequence can be finite or infinite.

geometric sequences

An individual takes out a \$20,000 car loan. The interest rate for the loan is 3%, compounded monthly. Assume a monthly payment of \$500. Define a_n to be the outstanding debt after n months. Since the interest rate describes the annual interest, the percentage increase each month is actually 3% / 12 = 0.25%. Thus, the multiplicative factor increase each month is 1.0025. The recurrence relation for $\{a_n\}$ is:

$$a_0 = \$20{,}000$$

$$a_n = (1.0025)\cdot a_{n-1} - 500 \qquad \text{for } n \geq 1$$

The first few values, to the nearest dollar, for the sequence $\{a_n\}$ are:

$a_0=\$20{,}000$ $\quad$ $a_1=\$19{,}550$

$a_2=\$19{,}099$ $\quad$ $a_3=\$18{,}647\cdots$

compound interest

You deposit \$10,000 in a savings account that yields 5% yearly interest. How much money will you have after 30 years?

$$b_n = b_{n-1} + rb_{n-1} = (1+r)^n b_0$$

Why?

arithmetic sequences

An arithmetic sequence is a sequence of real numbers the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

Each element is obtained by adding a constant d to the previous element; the first number is some arbitrary number (a)

Example:

3, 1, −1, −3, −5, −7,...

a = ? d = ?

recurrence relations

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Examples:

$a_0 = a$ (initial value)

$a_n = d + a_{n-1}$ *for* $n \geq 1$ (recurrence relation)

$a_0 = a$ (initial value)

$a_n = r \cdot a_{n-1}$ *for* $n \geq 1$ (recurrence relation)

write closed forms for a_n

recurrence relations

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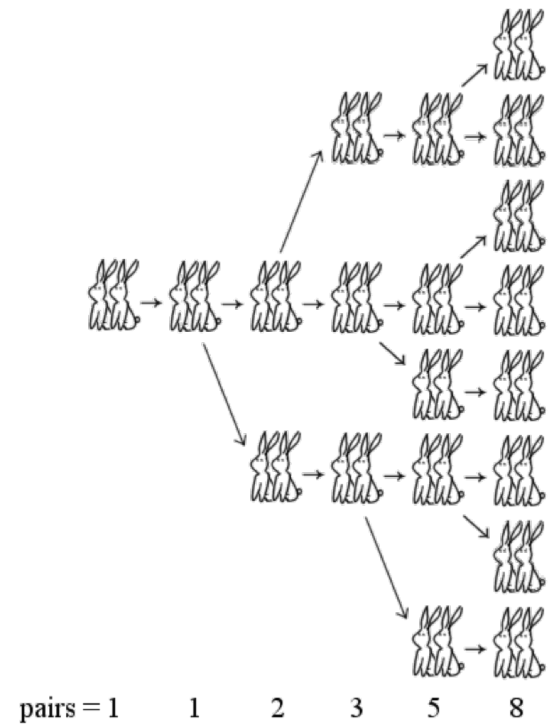
Example: the Fibonacci sequence

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2$$



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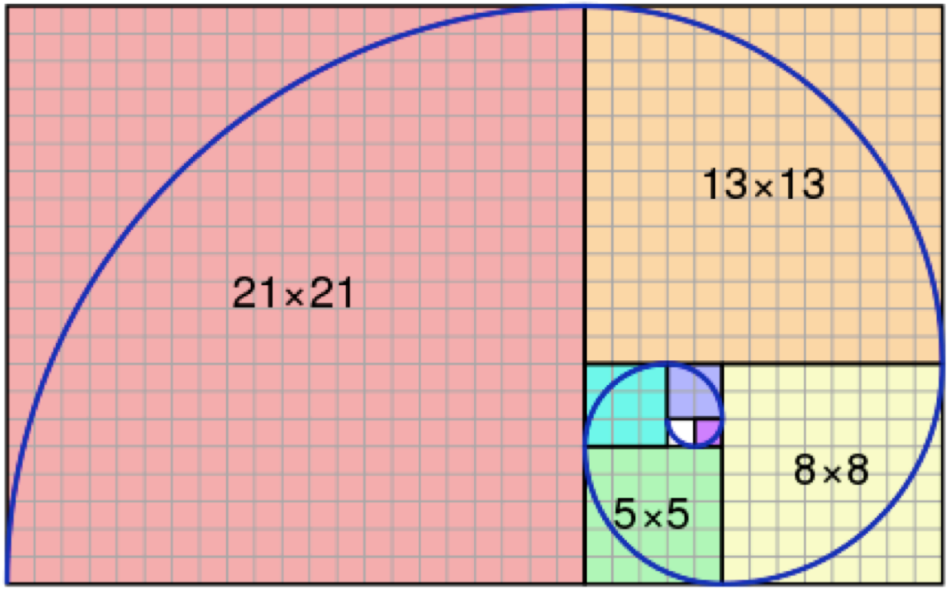

https://en.wikipedia.org/wiki/Fibonacci_number

summation notation

Consider a sequence $a_s, a_{s+1}, \ldots, a_t$

Notation to express the sum of the sequence:

$$a_s + a_{s+1} + \ldots + a_t = \sum_{i=s}^{t} a_i$$

upper limit

lower limit

Some useful sums:

arithmetic series: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

sum of squares: $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

geometric series: $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$

summation notation

Careful: Use of parentheses:

$$\sum_{j=1}^{t} (j^2 + 1) \quad \text{is not the same as} \quad \sum_{j=1}^{t} j^2 + 1$$