CS 220: Discrete Structures and their Applications

Mathematical Induction
6.4 - 6.6 in zybooks
Why induction?

Prove algorithm correctness (CS320 is full of it)

The inductive proof will sometimes point out an algorithmic solution to a problem

Strongly connected to recursion
Motivation

Show that any postage of $\geq 8¢$ can be obtained using 3¢ and 5¢ stamps.

First check for a few values:

- $8¢ = 3¢ + 5¢$
- $9¢ = 3¢ + 3¢ + 3¢$
- $10¢ = 5¢ + 5¢$
- $11¢ = 5¢ + 3¢ + 3¢$
- $12¢ = 3¢ + 3¢ + 3¢ + 3¢$

How to generalize this?
Let $n$ be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using triominoes, each covering three squares at a time.
Motivation

Prove that for every positive value of $n$,

$$1 + 2 + \ldots + n = n(n + 1)/2.$$
Motivation

Many mathematical statements have the form:
\[ \forall n \in \mathbb{N}, P(n) \]  
\( P(n) \): Logical predicate

Example: For every positive value of \( n \),
\[ 1 + 2 + \ldots + n = n(n + 1)/2. \]

Predicate - propositional function that depends on a variable, and has a truth value once the variable is assigned a value.

Mathematical induction is a proof technique for proving such statements.
Proving P(3)

Suppose we know:

1. P(1) and
2. \( P(n) \rightarrow P(n + 1) \ \forall n \geq 1. \)

Prove: P(3)

Proof:

1. P(1). [premise]
2. P(1) \( \rightarrow \) P(2). [specialization of premise]
3. P(2). [step 1, 2, & modus ponens]
4. P(2) \( \rightarrow \) P(3). [specialization of premise]
5. P(3). [step 3, 4, & modus ponens]

We can construct a proof for every finite value of \( n \).

Modus ponens: if \( p \) and \( p \rightarrow q \) then \( q \)
Example

**Theorem:** For every positive integer $n$,

$$
\sum_{j=1}^{n} j = \frac{n(n+1)}{2}
$$

**Proof.**

By induction on $n$.

**Base case:** $n = 1$.

When $n = 1$, the left side of the equation is $\sum_{j=1}^{1} j = 1$.

When $n = 1$, the right side of the equation is $1(1 + 1)/2 = 1$.

Therefore, $\sum_{j=1}^{1} j = \frac{1(1+1)}{2}$.
**Inductive step:** Suppose that for positive integer $k$, $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$, then we will show that

$$\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$$

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1)$$

by separating out the last term

$$= \frac{k(k+1)}{2} + (k+1)$$

by the inductive hypothesis

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

by algebra

Therefore, $\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$. ■
A Geometrical interpretation

1:  
2:  
3:  

Put these blocks, which represent numbers, together to form sums:

1 + 2 = 

1 + 2 + 3 =
A Geometrical interpretation

Area is $\frac{n^2}{2} + \frac{n}{2} = \frac{n(n + 1)}{2}$
The principle of mathematical induction

Let $P(n)$ be a statement that, for each natural number $n$, is either true or false.

To prove that $\forall n \in \mathbb{N}, P(n)$, it suffices to prove:

- $P(1)$ is true. \text{(basis step)}
- $\forall n \in \mathbb{N}, P(n) \rightarrow P(n + 1)$. \text{(inductive step)}

This is not magic.
It is a recipe for constructing a proof for an arbitrary $n \in \mathbb{N}$. 
the domino principle

If

the 1st domino falls over
and
the nth domino falls over implies that domino \((n + 1)\) falls over
then

domino \(n\) falls over for all \(n \in \mathbb{N}\).
proof by induction

3 steps:

- Prove $P(1)$. [the basis step]
- Assume $P(k)$ [the induction hypothesis]
- Using $P(k)$ prove $P(k + 1)$ [the inductive step]
Example

Show that any postage of \( \geq 8\)¢ can be obtained using 3¢ and 5¢ stamps.

Basis step:
\[
8¢ = 3¢ + 5¢
\]
Example

Let \( P(n) \) be the statement “\( n \) cents postage can be obtained using 3¢ and 5¢ stamps”.

Want to show that “\( P(k) \) is true” implies “\( P(k+1) \) is true” for all \( k \geq 8 \).

2 cases:

1) \( P(k) \) is true and the \( k \) cents contain at least one 5¢.

2) \( P(k) \) is true and the \( k \) cents do not contain any 5¢.
Case 1: $k$ cents contain at least one 5¢ stamp.

Case 2: $k$ cents do not contain any 5¢ stamp. Then there are at least three 3¢ stamp.
Arithmetic sequences

Sum of an arithmetic sequence:

\[ \sum_{j=0}^{n-1} (a + jd) = an + \frac{d(n-1)n}{2} \]

Proof: By induction on n

Base case: \( n=1 \)

Induction step:
Assume:

\[ \sum_{j=0}^{k-1} (a + jd) = ak + \frac{d(k-1)k}{2} \]

Need to prove:

\[ \sum_{j=0}^{k} (a + jd) = a(k + 1) + \frac{dk(k + 1)}{2} \]
Examples

Prove that \[ 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1 \]

Prove that for \( n \geq 4 \) \( 2^n < n! \)

Prove that \( n^3-n \) is divisible by 3 for every positive \( n \).

Prove that \[ 1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2 \]

Prove that a set with \( n \) elements has \( 2^n \) subsets.

Prove that \[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \] for \( n > 0 \)

Hint: \[ 2n^2+7n+6 = (n+2)(2n+3) \]

each time ask yourself

1. **BASE**
   What is the base case? Can I prove the base case?

2. **STEP**
   What is the hypothesis?
   **Obligation:** What do I need to prove the inductive step?
   How do I complete the inductive step?
Example

Prove that \( P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n > 0 \)

Base: \( n=1 \)  \( 1^2 = \frac{1 \cdot 2 \cdot 3}{6} \)

Hypothesis: \( P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \)

Obligation \( P(k) \rightarrow P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \)

Proof:

\[
1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \text{hypothesis} \\
k(k+1)(2k+1)/6 + (k+1)^2 = \frac{(k+1)(k(2k+1+6(k+1))}{6} = \\
\frac{(k+1)(2k^2+7k+6)}{6} = \text{Hint: } 2n^2+7n+6 = (n+2)(2n+3) \\
(k+1)(k+2)(2k+3)/6
\]
All horses have the same color

Base case: If there is only one horse, there is only one color.

Induction step: Assume as induction hypothesis that within any set of \( n \) horses, there is only one color. Now look at any set of \( n + 1 \) horses. Number them: 1, 2, 3, ..., \( n \), \( n + 1 \). Consider the sets \{1, 2, 3, ..., \( n \}\} and \{2, 3, 4, ..., \( n + 1 \}\}. Each is a set of only \( n \) horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all \( n+1 \) horses.

This is clearly wrong, but can you find the flaw?
NOT all horses have the same color

The step from
\[ k = 1 \quad \{1\} \]
to
\[ k = 2 \quad \{1,2\} \]

Fails: there is no intersection: \( \{1\} \cap \{2\} \) in a set of two horses, as was incorrectly used in the “proof”.
More induction examples

Let $n$ be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped triominoes, each covering three squares at a time.

First, show that

$$3 \mid 2^n \times 2^n - 1$$

(i.e. 3 divides $2^n \times 2^n - 1$)
Tiling with triominoes

Divide the board into four sub-boards:

Base case?
A bound on Fibonacci numbers

The Fibonacci sequence:
\[ f_0 = 0, f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2 \]

**Theorem:** \( f_n \leq 2^n \) for \( n \geq 0 \)
Strong induction

**Induction:**
- $P(1)$ is true.
- $\forall n \in \mathbb{N}, P(n) \to P(n + 1)$.
- Implies $\forall n \in \mathbb{N}, P(n)$

**Strong induction:**
- $P(1)$ is true.
- $\forall n \in \mathbb{N}, (P(1) \land P(2) \land ... \land P(n)) \to P(n + 1)$.
- Implies $\forall n \in \mathbb{N}, P(n)$
Example

Prove that all natural numbers $\geq 2$ can be represented as a product of primes.

Basis: $n=2$: 2 is a prime.
Example

Inductive step: show that \( n+1 \) can be represented as a product of primes.

- If \( n+1 \) is a prime: It can be represented as a product of 1 prime, itself.

- If \( n+1 \) is composite: Then, \( n + 1 = ab \), for some \( a,b < n + 1 \).
  - Therefore, \( a = p_1p_2 \ldots p_k \) & \( b = q_1q_2 \ldots q_l \), where the \( p_i \)'s & \( q_i \)'s are primes.
  - Represent \( n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l \).
Breaking chocolate

Theorem: Breaking up a chocolate bar with n “squares” into individual squares takes n-1 breaks.
(Break = dividing (sub) bar in 2 along a “break line”)

A full binary tree (sometimes proper binary tree or 2-tree) is a tree in which every node other than the leaves has two children.

Prove:
A full binary tree with n leaves has n-1 internal nodes

What is the relation with the chocolate bar?
Induction and the well ordering principle

The well-ordering principle:
any non-empty subset of the non-negative integers has a smallest element.

Surprising fact:
Well-ordering implies the principle of mathematical induction

Smallest element: base
Next element: step