CS 220: Discrete Structures and their Applications

Counting: the sum and product rules
zybooks 7.1 - 7.2

http://www.xkcd.com/936/
Counting is an important aspect of algorithm design and complexity analysis. We need to count:

- the number of loop iterations to establish the time complexity of our programs
- The number of elements of our arrays / lists /dictionaries to establish the space complexity of our programs
A simple counting problem

You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Possible answers:
A) $6 \times 10$
B) $6 + 10$
You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

We can express the set of all outfits as:

\{(s, p) \mid s \in \text{shirts and } p \in \text{pants}\}
You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

How would you write a program that prints out all the possible outfits? (Assume you have an array of shirts and an array of pants).
The **Cartesian product** of sets $A$ and $B$ is denoted by $A \times B$ and is defined as:

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

The product rule is simply a statement about the cardinality of a Cartesian product:

$$|A \times B| = |A| \times |B|$$
the product rule

The general statement of the product rule:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$
the product rule

Example: counting strings

Let $\Sigma$ is a set of characters (i.e. an alphabet); $\Sigma^n$ is the set of all strings of length $n$ whose characters come from the set $\Sigma$.

Applying the product rule:

$$|\Sigma^n| = |\Sigma \times \Sigma \times \cdots \times \Sigma| = |\Sigma| \cdot |\Sigma| \cdots |\Sigma| = |\Sigma|^n$$

Therefore, there are $2^n$ binary strings
the product rule

*Colorado assigns license plates numbers as three digits followed by three uppercase letters. How many license plate numbers are possible?*
the product rule

Colorado assigns license plates numbers as three digits followed by three uppercase letters. How many license plate numbers are possible?

A) $3^{10} \times 3^{26}$
B) $2 \times 3^{10} \times 3^{26}$
C) $10^3 \times 26^3$
How many bit strings of length 7 are there?
   A) $7 \times 6$  B) $7^2$  C) $2^7$

How many functions are there from a set with $m$ elements to a set with $n$ elements?
DNA is a long chain that contains one of four nucleotides (A,C,G,T). DNA can code for proteins that are chains of amino acids. There are 20 amino acids. How many nucleotides does it take to code for a single amino acid?

A) 2  
B) 3  
C) 4
Suppose you order a drink, and you can select either a hot drink or a cold drink. The hot drink selections are \{coffee, hot cocoa, tea\}. The cold drink selections are \{milk, orange juice\}. What is the total number of choices?

A) $3 + 2$
B) $3 \times 2$
C) $3! \times 2!$
another counting problem

Suppose you order a drink, and you can select either a hot drink or a cold drink. The hot drink selections are \{coffee, hot cocoa, tea\}. The cold drink selections are \{milk, orange juice\}. What is the total number of choices?

A) $3 + 2$
B) $3 \times 2$
C) $3! \times 2!$

The difference from the product rule: there is a single choice to be made.
The sum rule is also a statement about set theory: If two sets $A$ and $B$ are disjoint then
$$|A \cup B| = |A| + |B|$$
The general form of the sum rule:

Consider $n$ sets, $A_1, A_2, ..., A_n$. If the sets are mutually disjoint ($A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|$$
A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 4 possible projects. No project is on more than one list. How many possible projects are there to choose from?
The product and sum rule can be combined:

Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit. How many possible passwords are there?
The product and sum rule can be combined:

How many license plates can be made using either two or three uppercase letters followed by two or three digits?
The generalized product rule

Consider the following counting problem:

In a race with 20 runners there is a first place, a second place and a third place trophy. An outcome of the race is defined to be who wins each of the three trophies, i.e. 3 distinct runners. How many outcomes are possible?
The generalized product rule

Choosing sequences of items

Consider a set $S$ of sequences of $k$ items. Suppose there are:

- $n_1$ choices for the first item.
- For every possible choice for the first item, there are $n_2$ choices for the second item.
- For every possible choice for the first and second items, there are $n_3$ choices for the third item.
- ... 
- For every possible choice for the first $k-1$ items, there are $n_k$ choices for the $k$th item.

Then $|S| = n_1 \cdot n_2 \cdots n_k$. 
A family of four (2 parents and 2 kids) goes on a hiking trip. The trail is narrow and they must walk single file. How many ways can they walk with a parent in the front and a parent in the rear?
Consider the following definitions for sets of characters:

Digits = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \}

Letters = \{ a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z \}

Special characters = \{ *, &, $, # \}

Compute the number of passwords that satisfy the following constraints:

(a) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters.

(b) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character cannot be a special character.
Making Change

Goal. Given integer coin values, e.g.: \{1, 5, 10, 25\} compute in how many ways you can pay a certain amount:

Example: 29¢.

How many ways?

25, 1, 1, 1, 1
10, 10, 5, 1, 1, 1, 1
10, 5, 5, 5, 1, 1, 1, 1
10, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1

...
Given a coin set \( c = \{c_0, c_1, ..., c_{n-1}\} \) and an amount \( M \), how many different ways can \( M \) be paid?

Recursive solution: when \( d=n-1 \), given coin value \( c_d \), how many coins can I use?

- Example: for 56 cents, I can use 0, 1, or 2 quarters.

**Base:**
if \( d = 0 \), how many ways? (is there always a way?)

**Step:**
if \( d > 0 \), go through all possible uses of coin \( d \):
- at least how many \( c_d \) coins can be used
- and which problem then remains to be solved?

... 
- at most how many \( c_d \) coins can be used
- and which problem then remains to be solved?
Making Change

d=3: Quarters

d=2: Dimes

d=1: Nickles

d=0: Cents
One of your tasks in PA3 is to write a (recursive) function `mkCh()` that counts the number ways a certain amount of money can be paid with a coin set `{1,5,10,25}`
Given $n$ distinct elements, we want to group these into $k$ partitions. E.g. $n=3 \ \{b,c,d\}$ How many 2 out of 3 partitions are there?

Enumerate . . .

$\{b\} \ \{c,d\}$
$\{c\} \ \{b,d\}$
$\{d\} \ \{b,c\}$

there are 3 2-out-of-3 partitions
Counting the number of k-out-of-n partitions

Now \( n=4 \) \{a,b,c,d\}  How many 3-out-of-4 partitions are there?
Typical Divide and Conquer (hence recursive) approach: take element \( a \)
There are two possibilities:
1) either \( a \) is in its own partition or 2) not
   We can apply the sum rule
1) \( a \) is in its own partition, then there are 2 more partitions
   out of three elements \{b,c,d\}. We already solved how many 2 out
   of 3 partitions there are:
       \{b\} \{c,d\}
       \{c\} \{b,d\}
       \{d\} \{b,c\}
So for this case we get 3 solutions
   \{a\} \{b\} \{c,d\}
   \{a\} \{c\} \{b,d\}
   \{a\} \{d\} \{b,c\}
Second possibility

2) a is not in its own partition, then is it in 1 of the 3 3-out-of-3 partitions:

\{b\}  \{c\}  \{d\}

so in case 2) we have 3 possibilities

\{a,b\}  \{c\}  \{d\}
\{b\}  \{a,c\}  \{d\}
\{b\}  \{c\}  \{a,d\}

So in total there are 6 3-out-of-4 partitions
Do it yourself

How many 2-out-of-4 partitions are there?
Do it yourself

How many 2-out-of-4 partitions are there?

\{a,b,c,d\}

Option 1: a on its own

\{a\} \ {b,c,d\}

Option 2: a joins one of:

\{b\} \ {c,d\} \rightarrow \{a,b\} \ {c,d\}
\{b\} \ {a,c,d\}

\{c\} \ {b,d\} \rightarrow \{a,c\} \ {b,d\}
\{c\} \ {a,b,d\}

\{d\} \ {b,c\} \rightarrow \{a,d\} \ {b,c\}
\{d\} \ {a,b,c\}

So in total 7 (1 + 2*3) 2_out_of_4 groups
One of your tasks in PA3 is to write a (recursive) function
\texttt{partitions}(n,k)
that counts the number of k-out-of-n partitions