CS 220: Discrete Structures and their Applications

binary relations
zybooks 9.1-9.2
**binary relations**

A - set of students  
B - set of courses  
   R - pairs (a,b) such that student a is enrolled in course b  
   \[ R = \{ (\text{chris, cs220}), (\text{mike,cs520}),\ldots \} \]

A - set of cities  
B - set of US states  
   R - (a,b) such that city a is in state b  
   \[ R = \{ (\text{Denver, CO}), (\text{Laramie, WY}),\ldots \} \]
Definition: A binary relation between two sets $A$ and $B$ is a subset $R$ of $A \times B$.
Recall that $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$
For $a \in A$ and $b \in B$, the fact that $(a, b) \in R$ is denoted by $aRb$.

Example:
For $x \in \mathbb{R}$ and $y \in \mathbb{Z}$ define $xCy$ if $|x - y| \leq 1$
binary relations

a graphical representation of a relation

\[
\text{People} = \{ \text{Sue, Harry, Sam} \} \\
\text{Files} = \{ \text{FileA, FileB, FileC, FileD} \}
\]

Relation A: pAf if person p has access to file f

\[
A = \{ (\text{Sue, FileB}), (\text{Sue, FileC}), (\text{Sue, FileD}), \\
(\text{Harry, FileA}), (\text{Harry, FileD}) \}
\]
binary relations

the same binary relation can be represented as a matrix:

\[
\begin{array}{cccc}
\text{File A} & \text{File B} & \text{File C} & \text{File D} \\
\text{Sue} & 0 & 1 & 1 & 1 \\
\text{Harry} & 1 & 0 & 0 & 1 \\
\text{Sam} & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
A = \{(\text{Sue, File B}) (\text{Sue, File C}) (\text{Sue, File D}) (\text{Harry, File A}) (\text{Harry, File D}) \}
\]

A 2-d array of numbers with |A| rows and |B| columns. Each row corresponds to an element of A and each column corresponds to an element of B. For \( a \in A \) and \( b \in B \), there is a 1 in row \( a \), column \( b \), if \( aRb \) and 0 otherwise.
A binary relation from $A$ to $B$ is a subset of $A \times B$

Given sets $A$ and $B$ with sizes $n$ and $m$, the number of elements in $A \times B$ is $nm$, and the number of binary relations from $A$ to $B$ is $2^{nm}$

WHY?
functions as relations

A function $f$ from $A$ to $B$ assigns an element of $B$ to each element of $A$.

Difference between relations and functions?
A binary relation on a set $A$ is a subset of $A \times A$. The set $A$ is called the domain of the binary relation.

Graphical representation of a binary relation on a set:

$$A = \{a, b, c, d, e\}$$
$$R \subseteq A \times A$$
$$R = \{(a, b)(b, c)(e, c)(c, e)(d, a)(d, d)\}$$

self loop
Let \( A = \{1, 2, 3, 4\} \). Define a relation \( R \) on \( A \):
\[
R = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (4, 3)\}
\]

Can you find the mistakes in the following graphs and matrix representations of this relation?
binary relations on a set

Example: relations on the set of integers

\[ R_1 = \{(a,b) \mid a \leq b\} \]
\[ R_2 = \{(a,b) \mid a > b\} \]
\[ R_3 = \{(a,b) \mid a = b + 1\} \]
application of relations: knowledge graphs

Relations are a way of encoding knowledge.

A knowledge graph is a set of entities (Barak Obama, Hawaii, etc.), relations between those entities (<born_in>).

The relations are used to represent facts e.g. born_in(Barak Obama, Hawaii).

https://www.ambiverse.com/knowledge-graphs-encyclopaedias-for-machines/
properties of binary relations

Let R be a relation on a set A
The relation R is **reflexive** if for every \( x \in A \), \( xRx \).

**Example:** the less-or-equal to relation on the positive integers

The relation R is **anti-reflexive** if for every \( x \in A \), it is not true that \( xRx \).
properties of binary relations

Let R be a relation on a set A. The relation R is **transitive** if for every x, y, z ∈ A, xRy and yRz imply that xRz.

Example: the ancestor relation

\[ A = \{ a, b, c, d, e \} \]
properties of binary relations

Let $R$ be a relation on a set $A$.
The relation $R$ is **symmetric** if for every $x,y \in A$, $xRy$ implies that $yRx$.

Example:
$R = \{(a, b) : a, b \text{ are actors that have played in the same movie}\}$

The relation $R$ is **anti-symmetric** if for every $x,y \in A$, $xRy$ and $yRx$ imply that $x = y$.

$A = \{a, b, c, d, e\}$