CS 220: Discrete Structures and their Applications

counting by complement, inclusion exclusion, the pigeonhole principle

zybooks 7.8 - 7.10

example problem

How many 6-bit strings have at least one 0?

You can count them directly:

Number of 6-bit strings with at least one 0 =

Number of 6-bit strings with one 0
+ Number of 6-bit strings with two 0s
+ Number of 6-bit strings with three 0s
+ Number of 6-bit strings with four 0s
+ Number of 6-bit strings with five 0s
+ Number of 6-bit strings with six 0s

counting by complement

How many 6-bit strings have at least one 0?

Or you can use the complement rule:

Number of 6-bit strings with at one 0 =

Number of 6-bit strings

- Number of 6-bit strings with no 0s = $2^6 - 1$

The complement rule: Let P be a subset of a set S, then:

$$|P| = |S| - |\bar{P}|$$

example

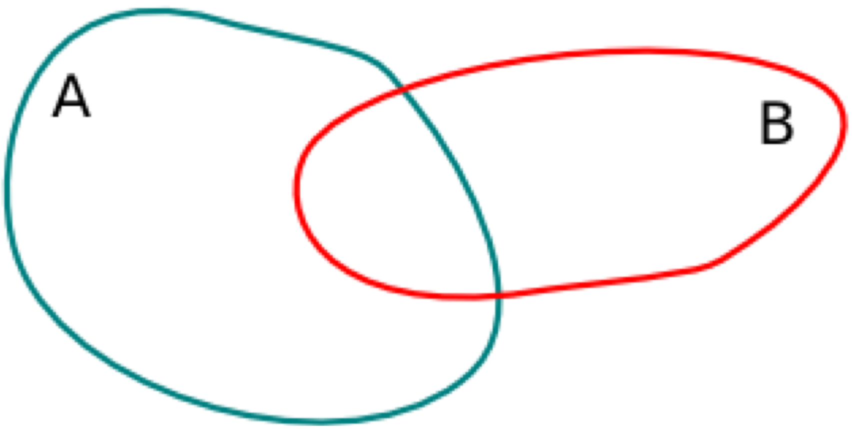
In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

- the bride must be next to the groom?
- The bride is not next to the groom?
- The bride is positioned somewhere to the left of the groom?

The inclusion exclusion principle

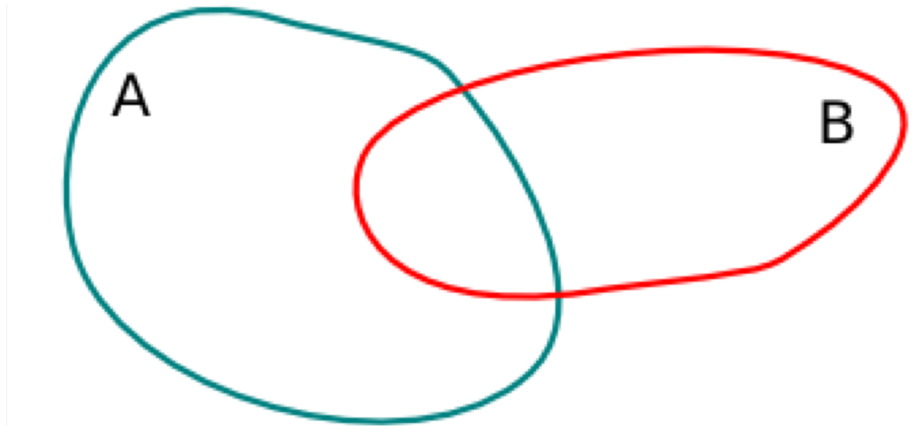
A more general statement than the sum rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example

How many numbers between 1 and 30 are divisible by 2 or 3?



The inclusion exclusion principle

How many bit strings of length eight start with a 1 or end with 00?

1 - - - - - - -   how many?

- - - - - - 0 0   how many?

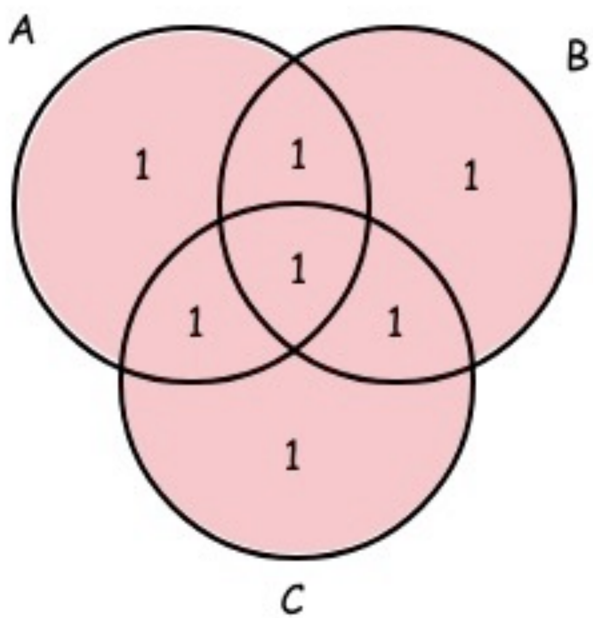
if I add these, how many have I counted twice?

inclusion exclusion with three sets

To compute the cardinality of the union of three sets:

Let A, B and C be three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



example

How many integers from 1 to 30 are divisible by 2, 3 or 5? $\boxed{22}$

A_2 = set of integers between 1 and 30 divisible by 2

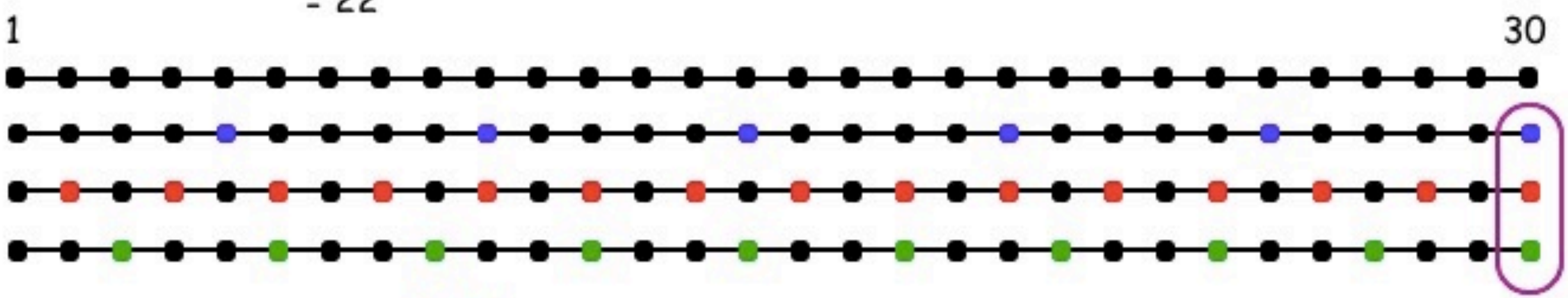
A_3 = set of integers between 1 and 30 divisible by 3

A_5 = set of integers between 1 and 30 divisible by 5

$$|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_5 \cap A_3| - |A_5 \cap A_2| + |A_5 \cap A_2 \cap A_3|$$

$$= 15 + 10 + 6 - 5 - 2 - 3 + 1$$

$$= 22$$



the inclusion exclusion principle

The general statement of the incusion-exclusion principle:

Let $A_1, A_2, \ldots, A_n$ be a set of n finite sets.

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{j=1}^{n} |A_j|$$

$$- \sum_{1 \le j < k \le n} |A_j \cap A_k|$$

$$+ \sum_{1 \le j < k < l \le n} |A_j \cap A_k \cap A_l|$$

$$\cdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Some advice about counting

Apply the multiplication rule if

- The elements to be counted can be obtained through a multistep selection process.
- Each step is performed in a fixed number of ways regardless of how preceding steps were performed.

Apply the sum rule if

- The set of elements to be counted can be broken up into disjoint subsets

Apply the inclusion/exclusion rule if

- It is simple to over-count and then to subtract duplicates

The pigeonhole principle

If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Image: http://en.wikipedia.org/wiki/File:TooManyPigeons.jpg

Examples

In a group of 367 people, there must be at least two with the same birthday

A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.

- How many socks must he take out to be sure that he has at least two socks of the same color?

  A) 13 B) 3 C) 12

Examples

In a group of 367 people, there must be at least two with the same birthday

A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.

- How many socks must he take out to be sure that he has at least two socks of the same color?
- How many socks must he take out to be sure that he has at least two black socks?

Examples

Show that if five different digits between 1 and 8 are selected, there must be at least one pair of these with a sum equal to 9.

ask yourself: what are the pigeon holes?

what are the pigeons?