CS 250: Foundation of Computer Systems [Binary Representations & Operations]

Powering up with Binary
All you have is a 0 and 1
But the fun's just begun
Simpler math operations
Multiplications simple as additions
Representing numbers on both sides of zero
Using two's complement, our unsung hero

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Frequently asked questions from the previous class survey

- Hexadecimal?
- Conventions: 0b or base-2, 0x or base-16, or oct/octal for base-8
- CPU cycles and the ALU:
- How does the CPU perform arithmetic operations?
- Does increasing the size of the memory (RAM) impact the speed of a system?
- Where is the Program Counter?
- What is the physical mechanism that allows re-writes on HDDs?
- What makes infinite loops useful? Do we code systems with infinite loops?
- Why not go to a 128-bit system?
- Memory concepts: registers, caches, main memory (RAM)

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Topics covered in this lecture

- Binary Representations
  - Properties of binary numbers
  - Operations on binary numbers
- Decimal to Binary
- Integers and word size implications
- Signed numbers

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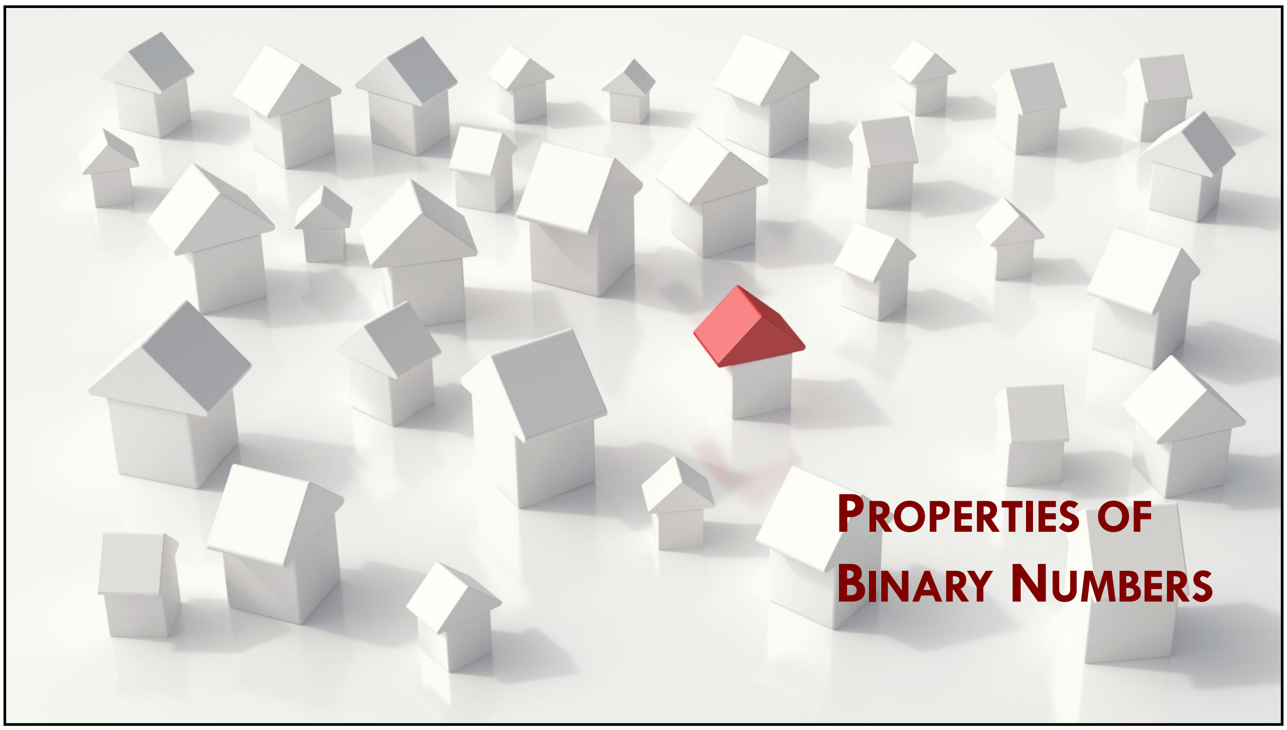

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Properties of binary numbers [4/8]

- Shifting all the bits in a number **to the left** by one position **multiplies** the binary value by 2
  - E.g.: 0b00000111 (value = 7)
    - Shift to the left (<<): 0b000001110 (value = $2^3 + 2^2 + 2^1 + 0 = 14$)
  - E.g.: 0b01010111 (value = 87)
    - Shift to the left (<<): 0b010101110
      - Value = $2^7 + 0 + 2^5 + 0 + 2^3 + 2^2 + 2^1 + 0 = 128 + 32 + 8 + 4 + 2 = 174$

Properties of binary numbers [5/8]

- Shifting all the bits of an unsigned binary number **to the right** by one position effectively **divides** that number by 2
  - This does not apply to signed integer values
  - Odd numbers are rounded down
- E.g.: 0b01010110 (value = 86)
  - Shift to the right (>>): 0b0101011~~0~~ = 0b0101011 = 43
- E.g.: 0b01010111 (value = 87)
  - Shift to the right (>>): 0b0101011~~1~~ = 0b0101011 = 43

Properties of binary numbers [6/8]

- Multiplying two *n*-bit binary values together may require *as many as* $2 \times n$ bits to hold the result
- Adding or subtracting two *n*-bit binary values never requires more than $n + 1$ bits to hold the result

Properties of binary numbers [7/8]

- *Incrementing* (adding 1 to) the largest unsigned binary value for a given number of bits always produces a value of 0
  - 0b11111111 = 0b00000000 (the last carryover of 1 **overflows**)
- *Decrementing* (subtracting 1 from) 0 always produces the largest unsigned binary value for a given number of bits

Number of unique combinations in a byte?

- Another way to think about this question is how many unique combinations of 0s and 1s can we make with our 8 bits?
- Let's first illustrate this with 4-bits

16 unique combinations of 0s and 1s in a 4-bit number, ranging in decimal value from 0 to 15

- We could determine the largest possible number that 4 bits can represent by setting all the bits to one, giving us 0b1111
  - That is 15 in decimal
- If we add 1 to account for representing 0, then we come to our total of 16

Binary	Decimal
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

Properties of binary numbers [8/8]

- In general, for n bits
  - The total number of unique combinations: 2^n
  - The largest possible number is $2^n - 1$

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DECIMAL TO BINARY

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Conversion from decimal to binary

- The binary number is constructed from **right to left**
- Step 1: Divide the decimal number by 2 and note down the remainder
- Step 2: Divide the obtained quotient by 2, and note remainder again
- Step 3: Repeat the above steps until you get **0** as the quotient
  - **Stopping criteria**

Example: Decimal number 23

- $23 \div 2 =$ **11** (Remainder 1)
- **11** $\div 2 =$ **5** (Remainder 1)
- **5** $\div 2 =$ **2** (Remainder 1)
- **2** $\div 2 =$ **1** (Remainder 0)
- **1** $\div 2 =$ **0** (Remainder 1)

Top to Bottom
is
Right to Left
{LSB to MSB}

- Binary Representation: **10111**
  - $2^4 + 2^2 + 2^1 + 2^0 = 16 + 4 + 2 + 1 = \mathbf{23}$

Example: Decimal number 99

- 99 ÷ 2 = **49** (Remainder 1)
- **49** ÷ 2 = **24** (Remainder 1)
- **24** ÷ 2 = **12** (Remainder 0)
- **12** ÷ 2 = **6** (Remainder 0)
- **6** ÷ 2 = **3** (Remainder 0)
- **3** ÷ 2 = **1** (Remainder 1)
- **1** ÷ 2 = **0** (Remainder 1)

Top to Bottom
is
Right to Left
{LSB to MSB}

- Binary Representation: **1100011**
  - $2^6 + 2^5 + 2^1 + 2^0$ = 64 + 32 + 2 + 1 **= 99**

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PREFIXES FOR LARGE COLLECTIONS

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Prefixes

- To more easily communicate the size of data, we use **prefixes** like giga- and mega-
- The International System of Units (SI), also known as the metric system, defines a set of standard prefixes
- These prefixes are used to describe anything that can be quantified, *not just bits*

Common SI Prefixes

Prefix name	Prefix symbol	Base-10 value	English word
Peta	P	10^{15}	quadrillion
Tera	T	10^{12}	trillion
Giga	G	10^{9}	billion
Mega	M	10^{6}	million
Kilo	K	10^{3}	thousand
centi	c	10^{-2}	hundredth
milli	m	10^{-3}	thousandth
micro	µ	10^{-6}	millionth
nano	n	10^{-9}	billionth
pico	p	10^{-12}	trillionth

These prefixes are used to describe anything that can be quantified, not just bits.

When dealing with bytes, most software is actually working in base 2, not base 10

- If your computer tells you that a file is 1MB in size, it is actually 1,048,576 bytes!
  - That is approximately one million, but not quite

Prefix name	Prefix symbol	Value	Base 2
Peta	P	1,125,899,906,842,624	2^{50}
Tera	T	1,099,511,627,776	2^{40}
Giga	G	1,073,741,824	2^{30}
Mega	M	1,048,576	2^{20}
Kilo	K	1024	2^{10}

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BINARY ARITHMETIC OPERATIONS

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Addition

- A pair of binary numbers can be added bitwise **from right to left**
  - Using the same decimal addition algorithm learned in elementary school
- First, we add the two rightmost bits, also called the least significant bits (LSB) of the two binary numbers
  - Next, add the resulting **carry** bit to the sum of the next pair of bits
- Continue this lockstep process ***until*** the two left most significant bits (MSB) are added

Adding binary numbers: **Rules**

- 0 + 0 = 0
- 0 + 1 = 1
- 1 + 0 = 1
- 1 + 1 = 0 with **carry**
- Carry + 0 + 0 = 1
- Carry + 0 + 1 = 0 with carry
- Carry + 1 + 0 = 0 with carry
- Carry + 1 + 1 = 1 with carry

Examples: Binary Addition

	0	0	0	1	
# 9		1	0	0	1
# 5		0	1	0	1
	0	1	1	1	0
	No Overflow				

CARRY	1	1	1	1		
		1	0	1	1	# 11
		0	1	1	1	# 7
	1	0	0	1	0	
	Overflow					

Addition of 9 and 5 = 8 + 4 + 2 = 14

Addition of 11 and 7 = 10010_2 = 18 (no truncation)
Addition of 11 and 7 = 0010_2 = 2 (with truncation to 4-bits)

Subtracting binary numbers: **Rules**

- 0 – 0 = 0
- 0 – 1 = 1 with a borrow
- 1 – 0 = 1
- 1 – 1 = 0

Subtraction: An example

		0	b		
# 5		0	1	0	1
# 3		0	0	1	1
		0	0	1	0

Borrow

Multiplication of binary numbers: Rules

- 0 × 0 = 0
- 0 × 1 = 0
- 1 × 0 = 0
- 1 × 1 = 1

Multiplication: An example

			1	0	1	0	#10
			0	1	0	1	# 5
			1	0	1	0	⟵ 1 x 1 0 1 0
							⟵ 0 x 1 0 1 0
	1	0	1	0			⟵ 1 x 1 0 1 0
	1	1	0	0	1	0	

Decimal representations:
10 x 5

0b110010 =
32 + 16+ 2 = 50

Division: An example

- 3456 ÷ 12 = 288
  - How? Let's take a look at a long-division example

Division in binary

- The basic algorithm is easier in binary because
  - At each step you don't have to guess how many times 12 goes into the remainder or multiply 12 by your guess to obtain the amount to subtract
  - At each step in the binary algorithm, **the divisor goes into the remainder exactly zero or one times**
- Let's take a look at: 0b11011 ÷ 0b11 = 0b1001
  - 27 ÷ 3 = 9

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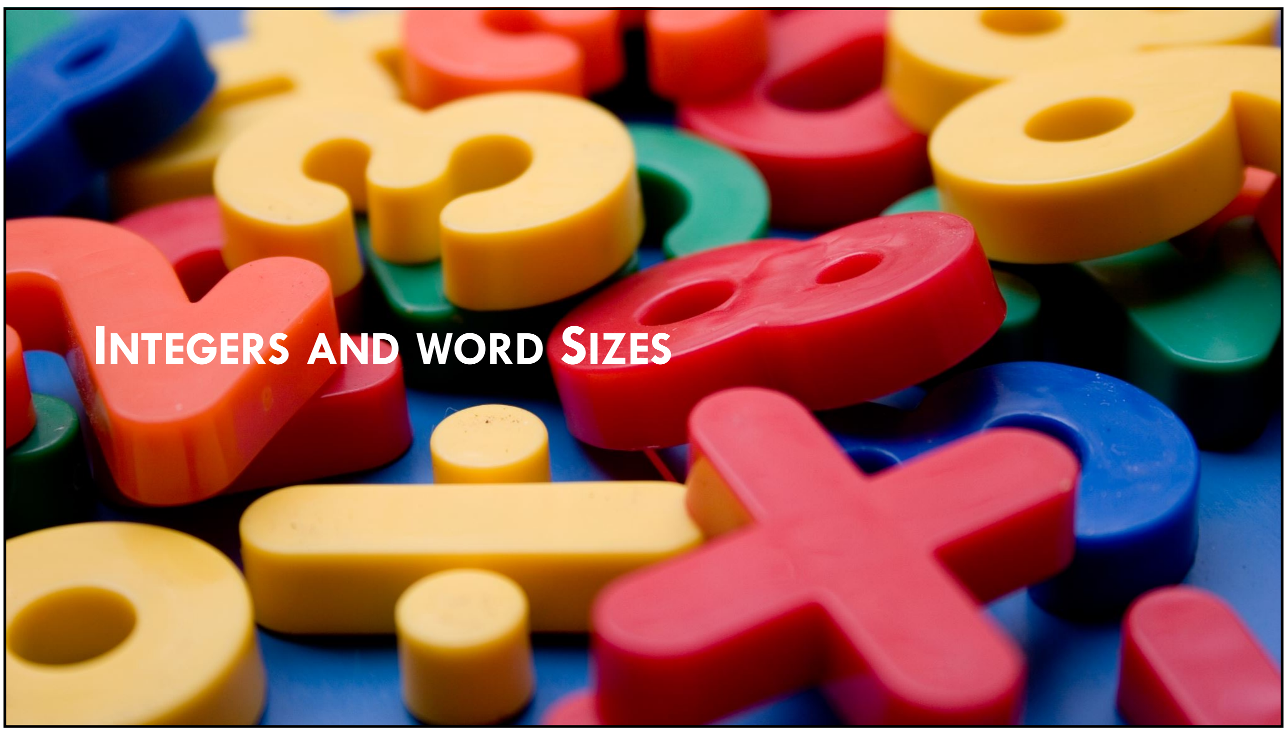

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Integer numbers are, of course, unbounded

- For any given number x:
  - There are integers that are less than x and integers that are greater than x
- Yet computers are *finite machines* that use a **fixed word size** for representing numbers
- **Word size** is a common hardware term used for specifying number of bits that computers use
  - For representing a basic chunk of information — in this case, integer values
  - Typically, 8-, 16-, 32-, or 64-bit registers are used for representing integers

Word size implications

- The **fixed word size** implies that there is a **limit** on the number of values that these registers can represent
- For example, suppose we use 8-bit registers for representing integers
  - This representation can code $2^8 = 256$ different things
  - If we wish to represent only nonnegative integers?
    - We can assign 00000000 for representing 0,
    - 00000001 for representing 1,
    - 00000010 for representing 2, 00000011 for representing 3,
    - All the way up to assigning 11111111 for representing 255

Word size implications

- In general, using n bits we can represent all the nonnegative integers ranging from 0 to $2^n - 1$
  - What about representing negative numbers using binary codes? Soon …

Representing numbers that are > maximal, or < minimal, values permitted by the fixed register size

- Every high-level language provides abstractions for handling numbers that are as large or as small as we can practically want
  - For e.g., java.math.BigInteger ($-2^{\text{Integer.MAX\_VALUE}}$ … $+2^{\text{Integer.MAX\_VALUE}}$)
  - Integer.MAX_VALUE = $2^{31}-1$ = 2,147,483,647
- These abstractions are typically implemented by **lashing together** as many n-bit registers as necessary for representing the numbers
- Since executing arithmetic and logical operations on multi-word numbers is a **slow affair**
  - It is recommended to use this practice only when the application requires processing extremely large or extremely small numbers

SIGNED NUMBERS

There's a sign on the wall, but she wants to be sure
'Cause you know sometimes words have two meanings
Stairway to Heaven, Jimmy Page / Robert Plant; Led Zeppelin

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Signed Numbers

- An n-bit binary system can code 2^n different things
- If we have to represent **signed** (+ and –) numbers in binary code, a solution is to split the available space into two subsets
  - One for representing nonnegative (+) numbers, and
  - The other for representing negative (–) numbers

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Desirable properties of signed number representations

- Ideally, the coding scheme should be chosen such that the introduction of signed numbers
  - Complicates the hardware implementation of arithmetic operations **as little as possible**
- This challenge has led to the development of several coding schemes for representing signed numbers in binary code

The solution used today in almost all computers?

- **Two's complement** method, also known as *radix complement*
- Example:
  - Consider a binary system that uses a word size of n bits
  - The two's complement binary code that represents negative x is taken to be the code that represents $2^n - x$
  - Representation of $-x$ ➔ $2^n - x$

Two's complement representation of 4-bit numbers

- Recall, in a n-bit number system
- $-x \rightarrow 2^n - x$
- For example, in a 4-bit binary system:
  - –7 is represented using the binary code associated with $2^4 - 7 = 9$
    - Which happens to be 1001

4-bit binary	Base-10	Derivation
0000	0	
0001	1	
0010	2	
0011	3	
0100	4	
0101	5	
0110	6	
0111	7	
1000	– 8	$2^4 - 8 = 16 - 8 = 8$
1001	– 7	$2^4 - 7 = 16 - 7 = 9$
1010	– 6	$2^4 - 6 = 16 - 6 = 10$
1011	– 5	$2^4 - 5 = 16 - 5 = 11$
1100	– 4	$2^4 - 4 = 16 - 4 = 12$
1101	– 3	$2^4 - 3 = 16 - 3 = 13$
1110	– 2	$2^4 - 2 = 16 - 2 = 14$
1111	– 1	$2^4 - 1 = 16 - 1 = 15$

Let's take a closer look at +7 and –7

- +7 = 0111
- –7 = 1001
- +7 –7 = 0000 (ignoring the overflow bit)

An n-bit binary system with two's complement representation has attractive properties

- The system codes 2^n signed numbers, ranging from -2^{n-1} to $(2^{n-1} - 1)$
- The code of any *nonnegative* number begins with a **0**
- The code of any *negative* number begins with a **1**
- To obtain the binary code of $-x$ from the binary code of x?
  - **Flip all the bits of x and add 1 to the result**

0000	0	
0001	1	
0010	2	
0011	3	
0100	4	
0101	5	
0110	6	
0111	7	
1000	−8	(16−8)
1001	−7	(16−7)
1010	−6	(16−6)
1011	−5	(16−5)
1100	−4	(16−4)
1101	−3	(16−3)
1110	−2	(16−2)
1111	−1	(16−1)

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Two's complement: binary code of $-x$ from the binary code of x?

- Flip all the bits of x and add 1 to the result

4-bit binary	Base-10	Flip the bits	Add 1	Base-10
0001	1	1110	1111	– 1
0010	2	1101	1110	– 2
0011	3	1100	1101	– 3
0100	4	1011	1100	– 4
0101	5	1010	1011	– 5
0110	6	1001	1010	– 6
0111	7	1000	1001	– 7
1000	8	0111	1000	– 8

0000	0	
0001	1	
0010	2	
0011	3	
0100	4	
0101	5	
0110	6	
0111	7	
1000	−8	(16−8)
1001	−7	(16−7)
1010	−6	(16−6)
1011	−5	(16−5)
1100	−4	(16−4)
1101	−3	(16−3)
1110	−2	(16−2)
1111	−1	(16−1)

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Another attractive feature of two's complement

- Subtraction is handled as a special case of addition
- To illustrate, consider 5–7 in our 4-bit binary number system
  - This 5 + (–7)
  - 0101 + 1001
  - = 1110
  - Which is –2

Implications of the two's complement method? [1/2]

- The two's complement method enables adding and subtracting signed numbers
  - Using nothing more than the hardware required for adding positive numbers!
- Every arithmetic operation, from multiplication to division to square root, can be implemented reductively using binary addition

Implications of the two's complement method? [2/2]

- A huge range of computer capabilities rides on top of binary addition
- The two's complement method *obviates the need for special hardware* for adding and subtracting signed numbers
- The two's complement method is one of the most **remarkable and unsung** heroes of applied computer science

ONE'S COMPLEMENT

Another useful concept to know … one's complement

- A not so **successful** attempt to represent signed numbers
- Get to negative numbers by taking positive numbers and flipping all the bits (i.e., 1 becomes 0 and 0 becomes 1)

One's complement

- Flipping each bit of 0111 (+7) yields 1000 (–7)
- Has **two** different representations for zero

Sign	2^2	2^1	2^0	Decimal
0	1	1	1	+7
0	1	1	0	+6
0	1	0	1	+5
0	1	0	0	+4
0	0	1	1	+3
0	0	1	0	+2
0	0	0	1	+1
0	0	0	0	+0
1	1	1	0	–1
1	1	0	1	–2
1	1	0	0	–3
1	0	1	1	–4
1	0	1	0	–5
1	0	0	1	–6
1	0	0	0	–7

One's complement: Addition is a little more complicated

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0	0	1	0	+2
1	1	1	0	−1
0	0	0	0	0
				1
				1

End-around carry

The contents of this slide-set are based on the following references

- Noam Nisan and Shimon Schocken. *The Elements of Computing Systems: Building a Modern Computer from First Principles*. 2nd Edition. ISBN-10/ ISBN-13: 0262539802 / 978-0262539807. MIT Press. [Preface, Chapter 1-2]
- Jonathan E. Steinhart. *The Secret Life of Programs: Understand Computers -- Craft Better Code*. ISBN-10/ ISBN-13 : 1593279701/ 978-1593279707. No Starch Press. [Chapter 1]
- Randall Hyde. Write Great Code, Volume 1, 2nd Edition: Understanding the Machine 2nd Edition. ASIN: B07VSC1K8Z. No Starch Press. 2020. [Chapter 2]
- Matthew Justice. *How Computers Really Work: A Hands-On Guide to the Inner Workings of the Machine*. ISBN-10/ISBN-13 : 1718500661/ 978-1718500662. No Starch Press. 2020. [Chapter 1]