Number Systems and Radix Conversion

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1 Introduction

These notes for CS 270 describe polynomial number systems. The material is not in the textbook, but will be required for PA1.

We humans are comfortable with numbers in the *decimal* system where each position has a *weight* which is a power of 10: units have a weight of 1 (10^0), ten's have 10 (10^1), etc. Even the fractional part after the decimal point have a weight that is a (negative) power of ten. So the number 143.25 has a value that is $1 * 10^2 + 4 * 10^1 + 3 * 10^0 + 2 * 10^{-1} + 5 * 10^{-2}$, i.e., $100 + 40 + 3 + \frac{2}{10} + \frac{5}{100}$. You can think of this as a polynomial with coefficients 1, 4, 3, 2, and 5 (i.e., the polynomial $1x^2 + 4x + 3 + 2x^{-1} + 5x^{-2}+$ evaluated at $x = 10$.

There is nothing special about 10 (just that humans evolved with ten fingers). We can use any *radix*, r, and write a number system with *digits* that range from 0 to $r - 1$. If our radix is larger than 10, we will need to invent "new digits". We will use the letters of the alphabet: the digit A represents 10, B is 11, K is 20, etc.

2 What is the value of a radix-r number?

Mathematically, a sequence of digits (the dot to the right of d_0 is called the *radix point* rather than the decimal point), $\ldots d_2 d_1 d_0 . d_{-1} d_{-2} \ldots$ represents a number n defined as follows

$$
\begin{aligned}
n &= \sum_i d_i r^i \\
&= \ldots + d_2 r^2 + d_1 r^1 + d_0 r^0 + d_{-1} r^{-1} + d_{-2} r^{-2} + \ldots
\end{aligned}
$$

Example What is $\frac{1}{3}$ in radix 3? **Answer:** 0.1.

Example That was (too) simple. Let's say we are working with radix 12. Then our digits are 0, 1, . . . 9, A and B. What does the number $(32A.B5)_{12}$ represent in decimal?
Answer: It's $3 * 12^2 + 2 * 12^1 + 10 * 12^0 + \frac{11}{12} + \frac{5}{12^2}$ which comes to 466.9514.

Notice that this answer involved fractions that has 12 and 144 in the denominator, se I rounded off to four digits after the decimal. When dealing with fractional numbers, sometimes we have an exact (terminating) representation while sometimes we don't. The interesting thing is that this depends on *the radix we are using*. For example, we all know that in the decimal system, the fraction $\frac{1}{3}$ is non-terminating: $0.3333\ldots$, but we just saw that in radix 3, it is simply 0.1. We'll return to fractional numbers in a bit. For now, let's just consider integers.

Exercise 1: What is $(48A6)_{12}$ in decimal? (work it out in the space below[1])

Post mortem: How did you do it? Did you use something like the following?

1. Calculate the different powers of 12: $12^3 = 1728$, $12^2 = 144$, 12, and 1.
2. Multiply each of these by the *values* corresponding to our digits: 4, 8, 10, and 6, to get, respectively, blah1, blah2, blah3, and blah4.
3. Finally add up these four blahs to get, blaaaah

OK, please do it, I'm not telling you the answer.

[1]Please do this without using a calculator. In this class, you should practice doing all the HW problems without calculators. No calculators are allowed in the exams.

How many multiplications did you do? If you did it the naive way, the first step itself, computing the powers of 12: 12^3, 12^2, 12^1, and 12^0, would be $2+1=3$ multiplications. And then, the multiplications by the the digits, 4, 5, A and 6, would require four more multiplications, for a total of 7. Not too bad, you say. But, for a k-digit number, this would be roughly $\frac{k^2}{2}$, a quadratic function of k. Of course, you can do it with far fewer. Since you need all the powers of 12 from 0 to k, you can do this by a sequence of only k multiplications: start with 1 and 12, and calculate the successive powers by simply multiplying the previous one by 12. In this way, for a k-digit number, you would do roughly $2k$ multiplications (once per digit in step 1 above, and once per digit in step 2).

There's an even simpler way, called Horner's rule and it will be useful if/when you have to do this under time pressure (e.g., in your exams). Given a k digit number $d_{k-1} \ldots d_0$, we work our way from left to right. Here's the algorithm.

1. Start with the value of the leftmost digit as your answer.

2. As long as there are more digits to the right of the current one, multiply the current answer by r and add the value of the next digit to it.

Example revisited: So for $(48A6)_{12}$, we would have the following successive values for answer:

- 4 — Initial step (digit 4)
- $4 * 12 + 8 = 56$ — digit 8
- $56 * 12 + 10 = 682$ — digit A (value 10)
- $682 * 12 + 6 = 8190$ — digit 6

So we see that finding out the value of a radix-r number is simply evaluating a polynomial whose coefficeints are the digits of our number, at r. Let's do a few exercises.

Exercise 2: What are the decimal values of $x_1 \dots x_3$ specified as:

(a) $x_1 = (DC95)_{14}$

(b) $x_2 = (1352)_6$

(c) $x_3 = (4421)_5$

3 Radix-r representation of a decimal number

We now want to go the other way, given x and r, what is the sequence of digits that represent x in radix r. Let's say that our answer is some sequence, $\ldots d_2 d_1 d_0$. Let's apply what we have learnt so far. If we convert this sequence to the number it represents, we get $x = \sum_i d_i r^i = \ldots d_2 r^2 + d_1 r^1 + d_0 r^0$. Since $r^0 = 1$, let's break the right hand side into two parts and simplify as follows

$$x = \left(\sum_{i>0} d_i r^i \right) + d_0 = r \left(\sum_{i>0} d_i r^{i-1} \right) + d_0$$

The simplification consisted of observing that each of the terms inside the summation had at least one factor r and pulling it outside the summation. The first part is now divisible by r, and the last term is less than r. In other words, if we divide x by r, the remainder will be d_0. This gives us the following procedure, that produces the required digits, but from *right to left*.

- Divide the number by r producing a quotient, a and remainder b. The least significant digit of the answer is b.
- If $a = 0$ we are done, otherwise repeat the process on the quotient, a.

Example: Let's convert 256 to base 6.

- $256 = 6 * 42 + 4$ $\quad d_0 = 4$
- $42 = 6 * 7 + 0$ $\quad d_1 = 0$
- $7 = 6 * 1 + 1$ $\quad d_2 = 1$
- $1 = 6 * 0 + 1$ $\quad d_3 = 1$

So the answer is 1104. Check it by converting $(1104)_6$ to base 10. Let's do a few exercises.

Exercise 3: What is the representation of the following numbers in the radix indicated?
(a) $8462 = (????)_{20}$?

(b) $732 = (????)_4$?

(c) $(8462)_9 = (????)_5$? [Hint: First convert from radix 9 to decimal and then to radix 5].

4 Now for fractions

So far, we saw how to convert a sequence of digits to the integer that it represents, and also how to convert an integer into the sequence of digits that represents it. Now we do the same for fractions.

Let's go back to our first example $(32A.B5)_{12}$, or rather just its fractional part, the number $(0.B5)_{12}$. We have already seen that its value is $\frac{11}{12}+\frac{5}{144}$, which we first simplify to $\frac{137}{144}$ before getting the rounded off value, 0.9514.

Consider another example, $(0.1111)_3$ whose value is $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$. Since division by 3 and its powers are going to give us non-terminating numbers, and let's say we wanted our answer up to four significant digits.

Now, we could do four separate divisions, to get four separate values, 0.3333, 0.1111, 0.0370, and 0.0123. Then we add them up to get 0.4937. On the other hand, if we first add up all the terms in the numerator, and do a single division we get, $\frac{27+9+3+1}{81}=\frac{40}{81}=0.4938$, which is different from the first answer. In the first method, rounding the four intermediate divisions to 4 digits lost us some precision, so the 0.4938 is the more accurate answer. This is the only thing you need to be careful about. Other than that converting a sequence of digits starting with a radix point to the decimal number that it represents is straightforward. Baically, you start from the rightmost digit, and work your way towards the radix point, and divide the accumulated answer at each step. What is the fractional part of $(x=2.2013)_4$.

- Initial answer is 0, imagine that the radix point is all the way to the right, i.e., the *fractional part* of $(x=22013.0)_4$ is 0.
- Next, push the radix point one place to the left and the fractional part becomes: (new-digit + old-answer)/4, i.e., $(3+0)/4=0.75$
- For the next digit we get: $(1+0.75)/4=0.4375$
- The next is a 0, so we get $0.4375/4=0.109375$
- Next we get $2.109375/4=0.52734375$, and we are done—the radix point is in the correct place.

Exercise 4 What is $(0.1011)_2$ in decimal?

Summary: So we saw three problems related to converting numbers from one radix to another

- To convert a radix-r representation (i.e., a sequence of symbols) into the integer it represents, we do repeated *multiplication*, working from the left to the right (*towards* the radix point).
- To convert an integer into a radix-r representation, we do repeated *division* and produce the answers right to left (*away from* the radix point).
- To convert a fractional radix-r representation to the number it represents, we do repeated *division* and work from right to left (*towards* the radix point.

The final problem is how to convert a decimal fraction to radix r,and as you may expect we do it by repeated multiplication, working away from the radix point. At at each step you look at the integer part of the answer (instead of the remainder). Let's work out an example.

Example What is $x = 0.90234375$ in radix 4, i.e., $0.90234375 = (????)_4$?
Answer: We multiply x by 4 to get 3.609375. So the first (leftmost) digit is 3 and we

repeat the process.

- $0.609375 * 4 = 2.4375$, so the next digit is 2 and we are left with 0.4375
- $0.4375 * 4 = 1.75$, so the next digit is 1 and we are left with 0.75
- $0.75 * 4 = 3.0$, so the next digit is 3 and we are left with 0, so we are done.

Exercises $0.4304 = (????)_5$?

Do this final exercise, it illustrates a subtle point [be patient]: $0.24 = (????)_4$?

5 Conclusion

You may be wondering why you are learning this stuff and what can you do with it.

It turns out that a special case is when $r = 2$, the binary system and its cousins when r is a power of two such as 4 (the octal system) or 16 (the hexadecimal system, which we will use extensively in this class.

The second important point is that this way of looking at numbers is a generalization. All the rules for arithmetic that you learnt in elementary school (addition, subtraction, cary/borrow, multiplication division) carry over to radix r numbers too. This gives you a powerful way to see how electrical circuits that do arithmetic operations can be built. We will soon see this in a couple of weeks.

Please try out a few examples of addition and subtraction in some non-standard radix, like 7, 12 etc.