Shortest Paths in graphs with arbitrary edge weights

Cormen et. al. 24.1
Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights $c_{vw}$, find shortest path from node $s$ to node $t$. Here we consider single target shortest paths problems and this time we allow zero and negative edge weights.
Dijkstra SSSP can fail with negative edges. Shortest path $s$ to $t$ is not $s \rightarrow t$ (length 1)

Shortest path $s$ to $t$ is $s \rightarrow u \rightarrow v \rightarrow t$ (length 0)

Greedy approach does NOT work here.
Shortest Paths: Failed Attempt

Re-weighting: what if we add a large enough value to each edge weight so all weights $> 0$?

Re-weighting. Adding a constant to every edge weight fails, as the shortest path does not need to have the minimum number of edges.
Negative Cost Cycles

Negative cost cycle.

Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; therefore we consider only graphs without negative cycles.

\[ c(W) < 0 \]

If there is no negative cycle the shortest path is simple (no nodes repeated).

What about 0 sum cycles?

We can also ignore those, as they do not change the path length.
Goal: SDSP, single destination shortest path
   Determine the shortest paths of all nodes to target node t

Observation: as we consider only graphs without negative cycles, and zero sum cycles do not add to the path length, we can ignore cycles in our algorithm altogether, searching for simple shortest paths.

Therefore, a shortest path does not repeat any node. Hence, any shortest path has at most n-1 = |V|-1 edges.

Objective: BF(G,s,t): for all nodes s and destination t, find the shortest simple path from s to t
A Dynamic Programming Approach

OPT(i, v) = length of shortest v-t path using at most i edges. We want to create a recurrence, i.e. express OPT(i,v) in some Opt(j,w) j<i:

- Case 1: path uses at most i-1 edges.
  - OPT(i, v) = OPT(i-1, v)

- Case 2: path uses at most i edges.
  - use some edge (v,w), and then the best w-t path using at most i-1 edges

\[
OPT(i, v) = \begin{cases} 
0 & \text{if } v = t, \text{ otherwise } \infty \\
\min \{ OPT(i-1, v), \min_{(v,w) \in E} \{ OPT(i-1, w) + c_{vw} \} \} & \text{if } i = 0 \\
\min \{ OPT(i-1, v), \min_{(v,w) \in E} \{ OPT(i-1, w) + c_{vw} \} \} & \text{otherwise}
\end{cases}
\]

What is the length of the optimal s-t path?

OPT(n-1, s), n = |V|

Notice: we prefix an edge to the paths we have already computed, and find the minimum path length.
Bellman Ford

\[ BF(G,s,t) \]
\[ n = |V| \]
array \( M[0..n-1,V] \)
\( M[0,t] = 0 \)
\( M[0,v] = \infty \) for all \( v \neq t \)
for \( i = 1 \) to \( n-1 \)
compute \( M[i,v] \) using the recurrence
return \( M[n-1,s] \)
foreach node $v$
foreach edge $(v, w) \in E$
    $M[i,v] \leftarrow \min(M[i-1,v], M[i-1,w]+c_{vw})$

$i=1$: find shortest path with $\leq 1$ edge

$M[1,t]=0$ (no edge $(t,w)$)
$M[1,a]=\min(\infty, M[0,t]+(-3))= -3$ (via $b$: $\infty$)
$M[1,b]=\min(\infty, M[0,b]=\infty, M[0,d]+(-1)=\infty-1=\infty, M[0,e]+(-2)=\infty-2=\infty)$
    $=\infty$
$M[1,c]=\min(\infty, M[0,t]+(3))= 3$ (via $b$: $\infty$)
$M[1,d]=\min(\infty, M[0,t]+(4))= 4$
$M[1,e]=\min(\infty, M[0,t]+(2))= 2$ (via $c$: $\infty$)
Bellman Ford

foreach node \( v \)
foreach edge \( (v, w) \in E \)
\[ M[i, v] \leftarrow \min(M[i-1, v], M[i-1, w] + c_{vw}) \]

- \( i=2 \): find shortest path with \( \leq 2 \) edges
  - \( M[2, t] = 0 \) (\( \not\exists \) edge \((t, w)\))
  - \( M[2, a] = \min(-3, M[1, t] + (-3)) = -3 \) (via \( b: \infty \))
  - \( M[2, b] = 0 \) (\( \text{bet:0, bdt:3} \))
  - \( M[2, c] = \min(3, M[1, t] + (3)) = 3 \) (via \( b: \infty \))
  - \( M[2, d] = \min(4, M[1, a] + (6)) = 3 \)
  - \( M[2, e] = \min(2, M[1, c] + (-3)) = 0 \)
Bellman Ford

For each column all edges are inspected:
$O(nm)$ time, $O(n^2)$ space. $n=|V|$, $m=|E|$
Practical Improvements

Space improvements.

- Since we only refer to the previous column, we can make a space improvement: instead of keeping the whole table we keep previous and current column. At the end of a sweep previous becomes current, and current gets updated in the next sweep.
- We can even use only ONE column, and update nodes using the latest value. This will sometimes get us an update one sweep earlier. Now we only need to maintain $M[v] = \text{length shortest } v-t \text{ path that we have found so far}$. Update is now:
  $$M[v] \leftarrow \min \{ M[v], M[w] + c_{vw} \}$$
- The role of $i$ is only as a counter
- Space complexity is now $O(n)$
// Uses only knowledge of neighboring nodes.
Shortest-Path(G, s, t) {
    foreach node v ∈ V:
        M[v] ← ∞
        successor[v] ← None
    M[t] = 0
    for i = 1 to n-1:
        foreach node v ∈ V
            foreach node w such that (v, w) ∈ E:
                if (M[v] < M[w] + c_{vw}):  
                    M[v] ← M[w] + c_{vw}
                    successor[v] ← w
            if no M[w] value changed in iteration i, stop.
Detecting Negative Cycles

Bellman-Ford can be used to detect negative cycles by running it one more iteration.

**Lemma.** If $\text{OPT}(n,v) = \text{OPT}(n-1,v)$ for all $v$, then there is no negative cycle on a path to $t$.

because if there is a negative cycle, we can keep bringing $\text{OPT}(i,v)$ down

**Lemma.** If $\text{OPT}(n,v) < \text{OPT}(n-1,v)$ for some node $v$, then there is a negative cycle on a path to $t$.

because, as argued before, without negative cycles the optimum path length has at most $n-1$ edges
Detecting Negative Cycles

**Theorem.** Can detect negative cost cycle in $O(mn)$ time.
- Add new node $t$ and connect all nodes to $t$ with 0-cost edge.
- Check if $\text{OPT}(n, v) = \text{OPT}(n-1, v)$ for target node $t$.
  - if so, then there are no negative cycles
  - if not, then there is a negative cycle
Do Bellman Ford for Negative cycles

![Diagram](image_url)

- Graph with nodes a, b, c, d and edges with weights 1 and -3.
- A cycle exists from a to b to c to d.

**Question:** Does the algorithm detect the negative cycle?
Do Bellman Ford for Negative cycles

Augment with $t$

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### SDSP problem 1

**Diagram:ICTURE**

**Table:FIGURE**
SDSP problem 2

Graph:

- Start node: S
  - Edges:
    - S to A: -2
    - S to B: 5
    - S to D: 4
    - S to T: 3

- Intermediate nodes:
  - A
  - B
  - C
  - D

- Edges:
  - A to C: 1
  - B to C: 2
  - B to D: 4
  - C to D: 4
  - C to T: 1
  - D to T: 1

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SDSP problem 2

![Graph with nodes S, A, B, C, D, and T connected by directed edges with weights.]

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