# Heaps & Heapsort

Charles Babbage (1864)

Analytic Engine (schematic)

# Heaps, heap sort and priority queues

**priority Queue**: data structure that maintains a set S of elements.

Each element v in S has a key **key(v)** that denotes the **priority** of v.

Priority Queue provides support for
  **adding**, **deleting** elements,
  **selection / extraction of**
    **smallest** (Min prioQ) or **largest** (Max prioQ) key element,
  **changing** key value.

# Applications

E.g. used in managing real time events where we want to get the earliest next event and events are added / deleted on the fly.

Sorting

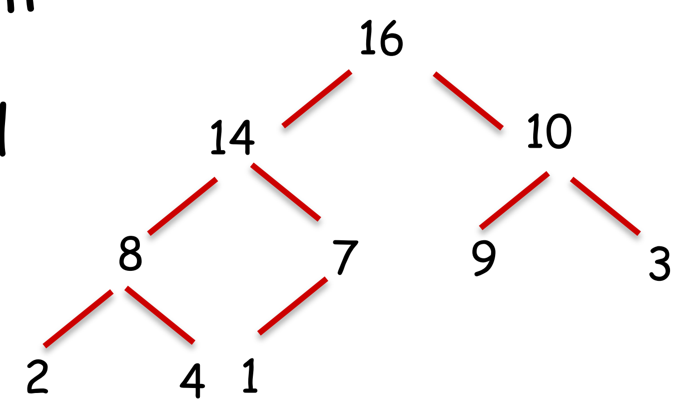
- build a prioQ
- Iteratively extract the smallest element

PrioQs can be implemented using **heaps**

# Heaps

Heap: **array** representation of a **complete** binary tree

- every level is completely filled except the bottom level: filled from left to right
- Can compute the index of parent and children: **WHY?**
  - parent(i) = floor((i-1)/2)
    leftChild(i)= 2i+1
    rightChild(i)=2(i+1)




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 14 | 10 | 8 | 7 | 9 | 3 | 2 | 4 | 1 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Max Heap property:
for all nodes i>0: **A[parent(i)] >= A[i]**
Max heaps have the max at the root

Min heaps have the min at the root

# Heapify(A,i,n)

To create a heap at index i, assuming left(i) and right(i) are heaps, **bubble A[i] down**: swap with max child until heap property holds

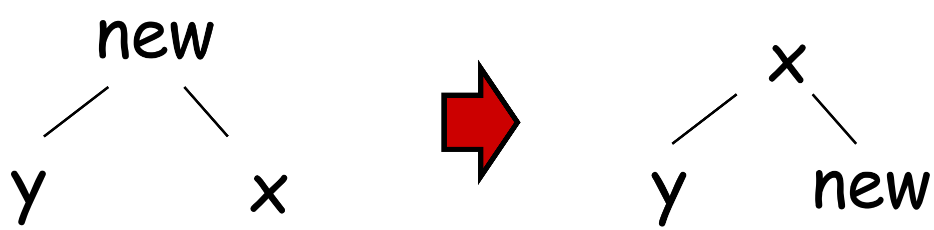
```
heapify(A,i,n):
# precondition
# n is the size of the heap
# tree left(i) and tree right(i) are heaps

    .......

# postcondition:  tree A[i] is a heap
```

# Swapping Down

Swapping down enforces (max) heap property at the swap location:




new<x and y<x:
swap(x,new)

x>y and x>new

**Are we done now?**

NO! When we have swapped we need to carry on checking whether new is in heap position. We stop when that is the case.

# Heap Extract

Heap extract:

Delete (and return) root

**Step 1:** replace root with last array element to keep completeness

**Step 2:** reinstate the heap property

Which element does **not** necessarily have the heap property?

How can it be fixed? **heapify the root**

Complexity? **O(log n)**

Swap **down:** swap with **maximum (maxheap), minimum (minheap)** child as necessary, until in place.

Sometimes called bubble down

Correctness based on the fact that we started with a heap, so the children of the root are heaps

# Heap Insert

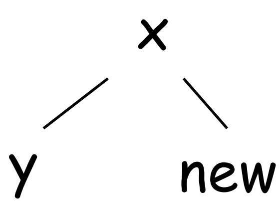
**Step 1**: put a new value into first open position (maintaining completeness), i.e. at the end of the array, but now we potentially violated the heap property, so:

**Step 2**: bubble up

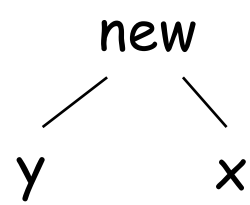
- **Re-enforcing the heap property**
- Swap with parent, if new value > parent, until in the right place.
- The heap property holds for the tree below the new value, when swapping up. **WHY? We only compared the new element to the parent, not to the sibling!**

# Swapping up

Swapping up enforces heap property for the sub tree below the new, inserted value:




if (new > x) swap(x,new)




x>y, therefore new > y

# Building a heap

heapify performs at most lg n swaps

**why? what is n?**

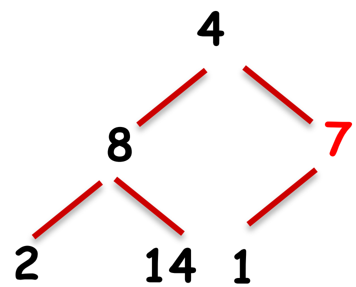
**buildheap:** builds a heap out of an array:

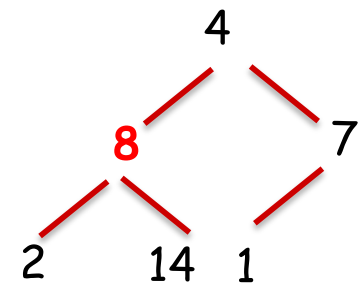
- the leaves are all heaps **WHY?**
- heapify **backwards** starting at **last internal** node

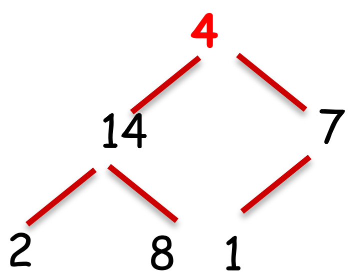
**WHY backwards?**
**WHY last internal node?**
**which node is that?**

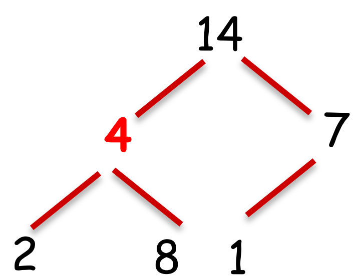
# LERT'S DO THE BUILDHEAP!

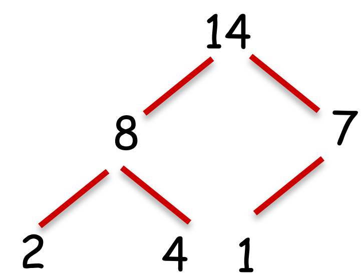
[4, 8, 7, 2, 14, 1]



# Complexity buildheap

Suggestions? ...

# Complexity buildheap

Initial thought: $O(n \lg n)$, but

- half of the heaps are height 0
- quarter are height 1
- only one is height $\log n$

It turns out that $O(n \lg n)$ is not tight!

# complexity buildheap

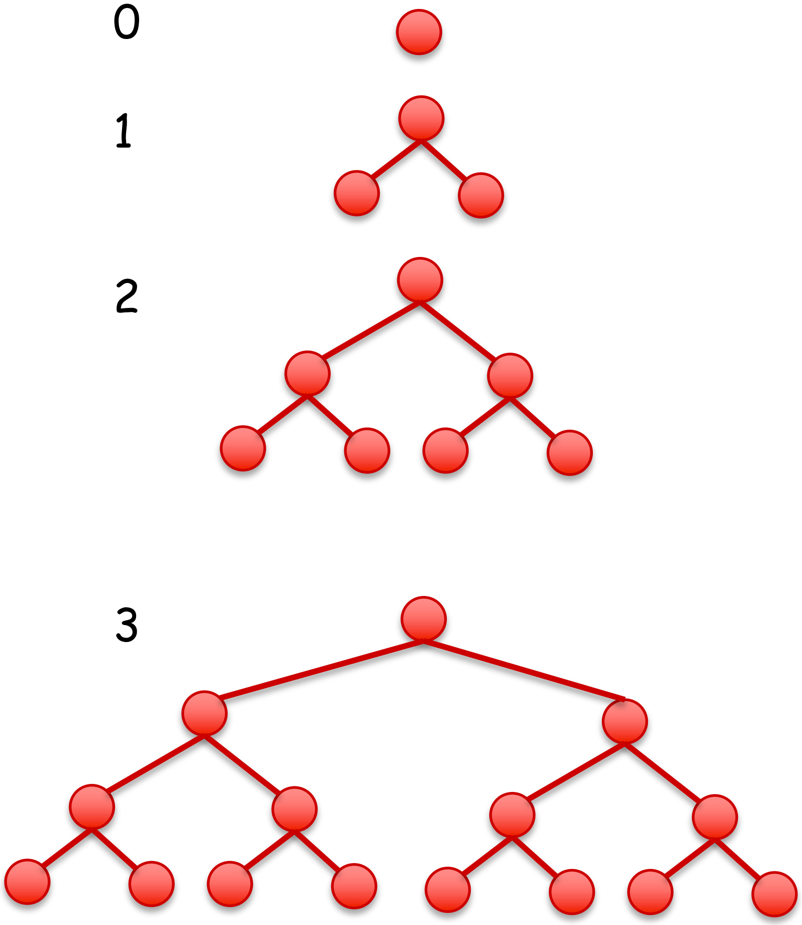
height max #swaps ?

0

1

2

3

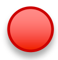


# complexity buildheap

height

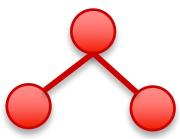
max #swaps, see a pattern?
(What kind of growth function do you expect ?)

0



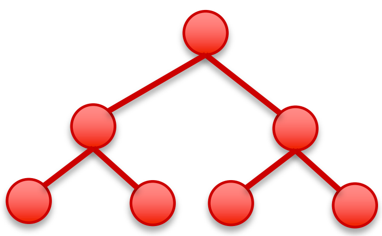
0

1



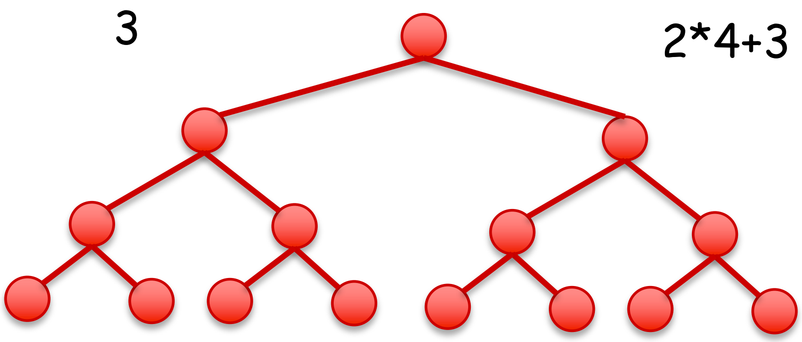
1

2



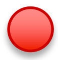
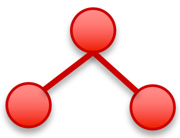
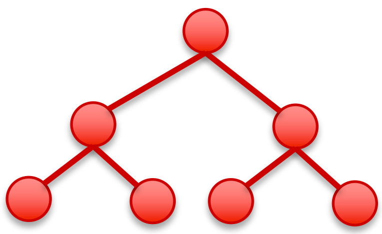
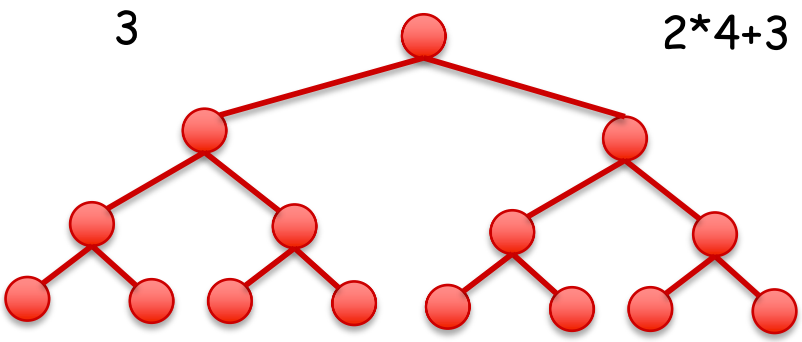
2*1+2 = 4

3

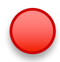
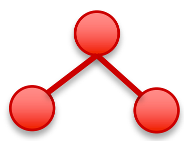
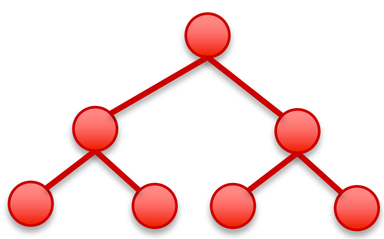
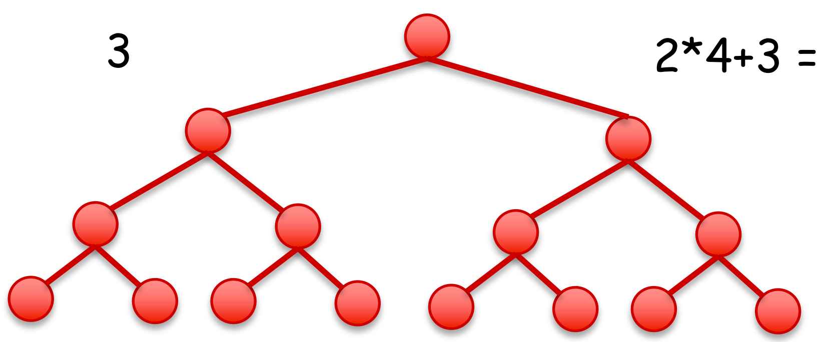


2*4+3 = 11

# complexity buildheap

| height | | max #swaps |
|---|---|---|
| 0 |  | $0 = 2^1-2$ |
| 1 |  | $1 = 2^2-3$ |
| 2 |  | $2*1+2 = 4 = 2^3-4$ |
| 3 |  | $2*4+3 = 11 = 2^4-5$ |

# complexity buildheap

| height | | max #swaps |
|---|---|---|
| 0 |  | $0 = 2^1-2$ |
| 1 |  | $1 = 2^2-3$ |
| 2 |  | $2*1+2 = 4 = 2^3-4$ |
| 3 |  | $2*4+3 = 11 = 2^4-5$ |

Conjecture:

height = h

max #swaps = $2^{h+1}-(h+2)$

Proof: induction

base?

step:

height = (h+1)

max #swaps:

$$2*(2^{h+1}-(h+2))+(h+1)$$
$$= 2^{h+2}-2h-4+h+1$$
$$= 2^{h+2}-(h+3)$$
$$= 2^{(h+1)+1}-((h+1)+2)$$

n nodes $\rightarrow \Theta(n)$ swaps

# Heapsort, complexity

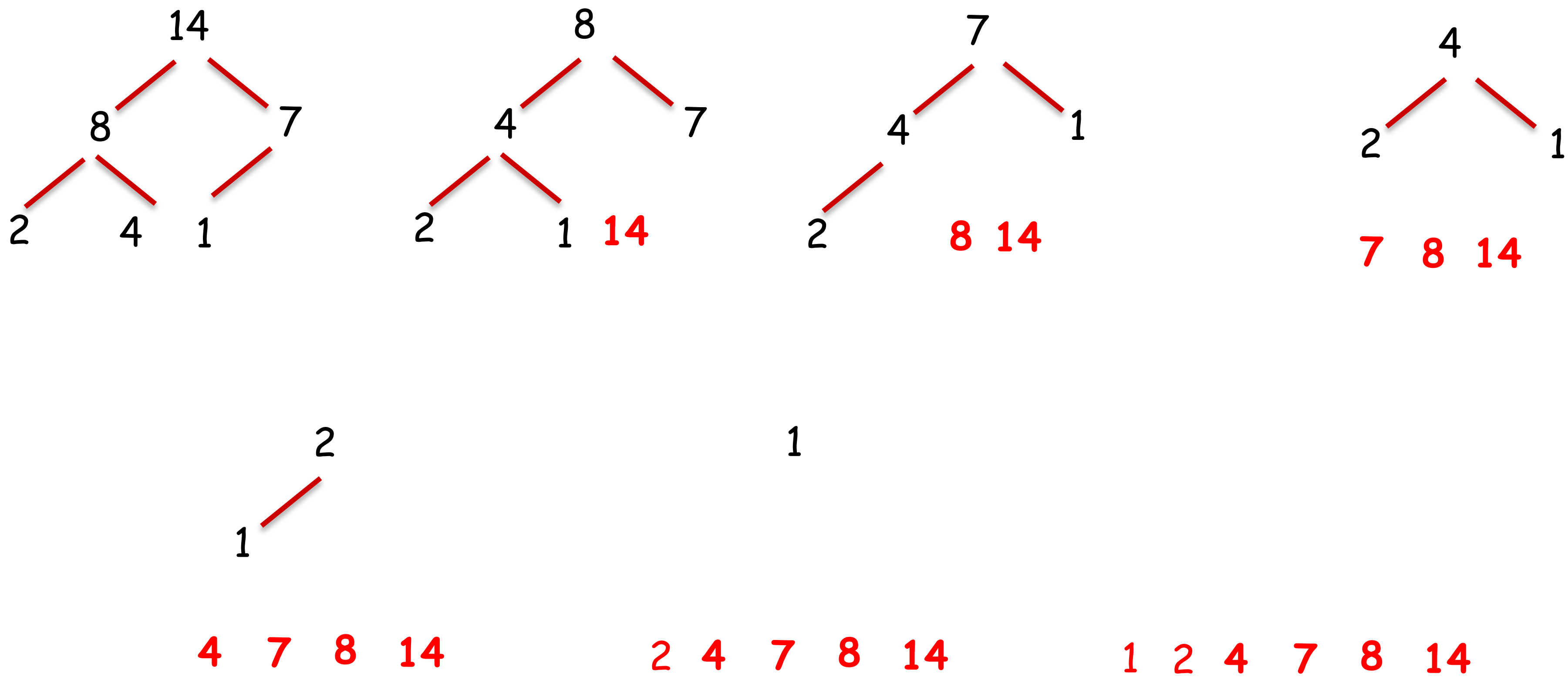
```
heapsort(A):
  buildheap(A)      # O( n )
  for i = n-1 downto 1 :                 #    O( ( n )
    # put max at end array

    # max is removed from heap
    n=n-1

    # reinstate heap property     #          * ( lg n)  )
```

- heapify: $\Theta(\lg n)$
- heapExtract: $\Theta(\lg n)$
- buildheap: $\Theta(n)$
- heapsort: $\Theta(n \lg n)$
- space: in place: $\Theta(n)$

# DO THE HEAPSORT!



# How **not** to heapExtract, heapInsert

```
# These "snail" implementations are NOT preserving the algorithm
# complexity of extractMin: log n and insert: log n and are therefore
# INCORRECT! from a complexity point of view  (even though they are
# functionally correct). Remember one of the goals of our course:
#     implementing the algorithms maintaining the analyzed complexity
# What are their complexities?

def snailExtractMin(A):
  n = len(A)
  if n == 0:
    return None
  min = A[0]
  A[0]=A[n-1]
  A.pop()
  buildHeap(A)   #  O(n)
  return min

def snailInsert(A,v):
   A.append(v)
   buildHeap(A)  #   O(n)
```

# Priority Queues

heaps can be used to implement priority queues:

- each value associated with a key
- max priority queue S has operations that maintain the heap property of S
  - max(S) returning max element
  - Extract-max(S) extracting and returning max element
  - increase key(S,x,k) increasing the key value of x
  - insert(S,x)
    - put x at end of S
    - bubble x up in place