Algorithm runtime analysis and computational tractability

As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - **Charles Babbage**

Charles Babbage (1864)  
Analytic Engine (schematic)
Time Complexity of an Algorithm

How do we measure the complexity (time, space requirements) of an algorithm.

The size of the problem: an integer n
- # inputs (for sorting problem)
- # digits of input (for the primality problem)
- sometimes more than one integer

We want to characterize the running time of an algorithm for increasing problem sizes by a function $T(n)$
Units of time

1 microsecond?

1 machine instruction?

# of code fragments that take constant time?
Units of time

1 microsecond?

no, too specific and machine dependent

1 machine instruction?

no, still too specific and machine dependent

# of code fragments that take constant time?

yes

# what kind of instructions take constant time?

arithmetic op, memory access
unit of space

bit?

int?
bit?

very detailed but sometimes necessary

int?

nicer, but dangerous: we can code a whole program or array (or disk) in one arbitrary int, so we have to be careful with space analysis (take value ranges into account when needed). Better to think in terms of machine words

i.e. fixed size, e.g. 64, collections of bits
Worst-Case Analysis

Worst case running time.

A bound on largest possible running time of algorithm on inputs of size n.
- Generally captures efficiency in practice, but can be an overestimate.

Same for worst case space complexity
Average case

**Average case running time.** A bound on the average running time of algorithm on random inputs as a function of input size $n$. In other words: the expected number of steps an algorithm takes.

$$\sum_{i \in I_n} P_i C_i$$

- $P_i$: probability input $i$ occurs
- $C_i$: complexity given input $i$
- $I_n$: all possible inputs of size $n$

- Hard to model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.
- Often hard to compute.
Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.

<table>
<thead>
<tr>
<th>n</th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>4 sec</td>
</tr>
<tr>
<td>30</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>18 min</td>
<td>$10^{25}$ years</td>
</tr>
<tr>
<td>50</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>100</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>$10^{17}$ years</td>
<td>very long</td>
</tr>
<tr>
<td>1,000</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>10,000</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>100,000</td>
<td>&lt; 1 sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>1,000,000</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>
A definition of tractability: Polynomial-Time

**Brute force.** For many problems, there is a natural brute force search algorithm that checks every possible solution.
- Typically takes $2^n$ time or worse for inputs of size $n$.
- Unacceptable in practice.
  - Permutations, TSP

An algorithm is said to be polynomial if there exist constants $c > 0$ and $d > 0$ such that on every input of size $n$, its running time is bounded by $c n^d$ steps.

- What about an $n \log n$ algorithm?
Worst-Case Polynomial-Time

On the one hand:

■ Possible objection: Although $6.02 \times 10^{23} \times n^{20}$ is technically poly-time, it would be useless in practice.
■ In practice, the poly-time algorithms that people develop typically have low constants and low exponents.
■ Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

On the other:

■ Some exponential-time (or worse) algorithms are widely used because the worst-case (exponential) instances seem to be rare.
  • simplex method solving linear programming problems
Comparing algorithm running times

Suppose that algorithm A has a running time bounded by

\[ T(n) = 1.62 \, n^2 + 3.5 \, n + 8 \]

- It is hard to get this kind of exact statement
  - It is probably machine dependent

- There is more detail than is useful

- We want to quantify running time in a way that will allow us to identify broad classes of algorithms

- I.e., we only care about Orders of Magnitude
  - in this case: \( T(n) = O(n^2) \)
Asymptotic Growth Rates
Upper bounds

T(n) is \( O(f(n)) \) if there exist constants \( c > 0 \) and \( n_0 \geq 0 \) such that

\[
\text{for all } n \geq n_0 : \ T(n) \leq c \cdot f(n)
\]

Example: \( T(n) = 32n^2 + 16n + 32 \).
- \( T(n) \) is \( O(n^2) \)
- BUT ALSO: \( T(n) \) is \( O(n^3) \), \( T(n) \) is \( O(2^n) \).

There are many possible upper bounds for one function! We always look for a tight bound \( \Theta(f(n)) \) later, but it is not always easy to establish
Expressing Lower Bounds

Big O Doesn't always express what we want:

Any comparison-based sorting algorithm requires at least $c(n \log n)$ comparisons, for some constant $c$.

- Use $\Omega$ for lower bounds.

$T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0 : T(n) \geq c \cdot f(n)$

Example: $T(n) = 32n^2 + 16n + 32$.

- $T(n)$ is $\Omega(n^2)$, $\Omega(n)$. 
Tight Bounds

$T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.

**Example:** $T(n) = 32n^2 + 17n + 32$.
- $T(n)$ is $\Theta(n^2)$.

If we show that the running time of an algorithm is $\Theta(f(n))$, we have closed the problem and found a bound for the problem and its algorithm solving it.

- **excursion:** heap sort and priority queues
F(n) is $O(G(n))$

F(n) is $\Omega(H(n))$

if $G(n) = c.H(n)$
then F(n) is $\Theta(G(n))$
Priority Queue: data structure that maintains a set $S$ of elements.

Each element $v$ in $S$ has a key $\text{key}(v)$ that denotes the priority of $v$.

Priority Queue provides support for
  - adding, deleting elements,
  - selection / extraction of
    - smallest $(\text{Min prioQ})$ / largest $(\text{Max prioQ})$ key element,
  - changing key value.
Applications

E.g. used in managing real time events where we want to get the earliest next event and events are added / deleted on the fly.

**Sorting**
- build a prioQ
- Iteratively extract the smallest element

PrioQs can be implemented using **heaps**
Heaps

Heap: array representation of a complete binary tree
- every level is completely filled except the bottom level: filled from left to right
- Can compute the index of parent and children:
  - parent(i) = floor((i-1)/2)
  - leftChild(i)= 2i+1
  - rightChild(i)=2(i+1)

Max Heap property:
for all nodes i>0: \( A[\text{parent}(i)] \geq A[i] \)

Min heaps have the min at the root
Heapify(A, i, n)

To create a heap at index i, assuming left(i) and right(i) are heaps, bubble A[i] down: swap with max child until heap property holds

heapify(A, i, n):
  # precondition
  # n is the size of the heap
  # tree left(i) and tree right(i) are heaps

      .......

  # postcondition: tree A[i] is a heap
Heap Extract

Heap extract:
Delete (and return) root

**Step 1:** replace root with last array element to keep completeness

**Step 2:** reinstate the heap property
Which element does not necessarily have the heap property?
How can it be fixed? Complexity?

heapify the root $O(\log n)$

Bubble **down:** swap with maximum (maxheap), minimum (minheap) child as necessary, until in place

Correctness based on the fact that we started with a heap, so the children of the root are heaps
Swapping down enforces (max) heap property at the swap location:

new < x and y < x: swap(x, new) → x > y and x > new
Heap Insert

Step 1: put a new value into first open position (maintaining completeness), i.e. at the end of the array

but now we potentially violated the heap property, so:

Step 2: bubble up

- Re-enforcing the heap property
  - Swap with parent, if new value > parent, until in the right place.
  - The heap property holds for the tree below the new value, when swapping up
Building a heap

heapify performs at most $\lg n$ swaps

why? what is $n$?

building a heap out of an array:

- the leaves are all heaps
- heapify backwards starting at last internal node

WHY backwards?
Suggestions? ...
Initial thought: $O(n \log n)$, but

half of the heaps are height 1
quarter are height 2
only one is height $\log n$

It turns out that $O(n \log n)$ is not tight!
Complexity buildheap

<table>
<thead>
<tr>
<th>height</th>
<th>max #swaps?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
complexity buildheap

max #swaps, see a pattern?
(What kind of growth function do you expect?)

<table>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2*1+2 = 4</td>
</tr>
<tr>
<td>3</td>
<td>2*4+3 = 11</td>
</tr>
</tbody>
</table>
height | max #swaps
--- | ---
0 | $0 = 2^1 - 2$
1 | $1 = 2^2 - 3$
2 | $2*1+2 = 4 = 2^3 - 4$
3 | $2*4+3 = 11 = 2^4 - 5$
complexity buildheap

<table>
<thead>
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<th>max #swaps</th>
</tr>
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<tr>
<td>0</td>
<td>0 = 2^1-2</td>
</tr>
<tr>
<td>1</td>
<td>1 = 2^2-3</td>
</tr>
<tr>
<td>2</td>
<td>2*1+2 = 4 = 2^3-4</td>
</tr>
<tr>
<td>3</td>
<td>2*4+3 = 11 = 2^4-5</td>
</tr>
</tbody>
</table>

Conjecture:
height = h
max #swaps = 2^{h+1}-(h+2)

Proof: induction
base?
step:
height = (h+1)
max #swaps:
2*(2^{h+1}-(h+2))+(h+1)
= 2^{h+2}-2h-4+h+1
= 2^{h+2}-(h+3)
= 2^{(h+1)+1}-(((h+1)+2)

n nodes $\rightarrow \Theta(n)$ swaps
See it the Master theorem way

\[ T(n) = 2*T(n/2) + \lg n \]

**Master theorem**

\[ \Theta(n^{\lg_2 2}) = \Theta(n) \]
Heapsort, complexity

heapsort(A):
    buildheap(A)
    for i = n-1 downto 1 :
        # put max at end array

        # max is removed from heap
        n=n-1

        # reinstate heap property

- buildheap: \(\Theta(n)\)
- heapsort: \(\Theta(n \log n)\)

- space: in place: \(\Theta(n)\)
DO IT, DO IT!
Priority Queues

heaps can be used to implement priority queues:

- each value associated with a key
- max priority queue \( S \) has operations that maintain the heap property of \( S \)
  - \( \text{max}(S) \) returning max element
  - \( \text{Extract-max}(S) \) extracting and returning max element
  - \( \text{increase key}(S, x, k) \) increasing the key value of \( x \)
  - \( \text{insert}(S, x) \)
    - put \( x \) at end of \( S \)
    - bubble \( x \) (see Increase-key)
Transitivity.
- If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.
- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = \Theta(h)$ then $f + g = \Theta(h)$.
Asymptotic Bounds for Some Common Functions

**Polynomials.**  $a_0 + a_1 n + \ldots + a_d n^d$ is $O(n^d)$ if $a_d > 0$.

**Polynomial time.** Running time is $O(n^d)$ for some constant $d$.

**Logarithms.**  $\log_a n$ is $O(\log_b n)$ for any constants $a, b > 0$.

- can avoid specifying the base
- for every $x > 0$, $\log n$ is $O(n^x)$.

**Combinations.** Merge sort, Heap sort $O(n \log n)$

**Exponentials.**  For every $r > 1$ and every $d > 0$, $n^d$ is $O(r^n)$.

- exponential grows faster than any polynomial
Problems have lower bounds

A problem has a lower bound $\Omega(f(n))$, which means:

Any algorithm solving this problem takes at least $\Omega(f(n))$ steps.

We can often show that an algorithm has to "touch" all elements of a data structure, or produce a certain sized output. This then gives rise to an easy lower bound.

Sometimes we can prove better (higher, stronger) lower bounds (eg Searching and Sorting (cs420)).
Problems have lower bounds, algorithms have upper bounds. A **closed problem** has a lower bound $\Omega(f(n))$ and at least one algorithm with upper bound $O(f(n))$

- Example: sorting is $\Omega(n \log n)$ and there are $O(n \log n)$ sorting algorithms.
  
  To show this, we need to reason about lower bounds of problems (cs420)

An **open problem** has lower bound $< $ upper bound

- Example: matrix multiplication (multiply two $n \times n$ matrices).
  
  - Takes $\Omega(n^2)$ why?
  - Naïve algorithm: $O(n^3)$
  - Coppersmith-Winograd algorithm: $O(n^{2.376})$
A Survey of Common Running Times
Constant time: $O(1)$

A single line of code that involves “simple” expressions, e.g.:

- Arithmetical operations (+,-,*,/) for fixed size inputs
- assignments ($x = \text{simple expression}$)
- conditionals with simple sub-expressions
- function calls (excluding the time spent in the called function)
Logarithmic time

Example of a problem with $O(\log(n))$ bound: binary search

How did we get that bound?
log(n) and algorithms

When in each step of an algorithm we halve the size of the problem then it takes $\log_2 n$ steps to get to the base case.

We often use $\log(n)$ when we should use $\text{floor}(\log(n))$. That's OK since $\text{floor}(\log(n))$ is $\Theta(\log(n))$.

Similarly, if we divide a problem into $k$ parts the number of steps is $\log_k n$. For the purposes of big-O analysis it doesn't matter since $\log_a n$ is $O(\log_b n)$ WHY?
Logarithms

**definition:**

\[ b^x = a \Rightarrow x = \log_b a, \ \text{eg} \ 2^3=8, \ \log_2 8=3 \]

\[ b^{\log_b a} = a \quad \log_b b = 1 \quad \log 1 = 0 \]

- \( \log(x \cdot y) = \log x + \log y \) because \( b^x b^y = b^{x+y} \)
- \( \log(x/y) = \log x - \log y \)
- \( \log x^a = a \log x \)
- \( \log x \) is a 1-to-1 monotonically (slow) growing function
- \( \log x = \log y \iff x = y \)

- \( \log_a x = \log_b x / \log_b a \)
- \( y^{\log x} = x^{\log y} \)
\[ \log_a x = \log_b x / \log_b a \]

\[ b^{\log_b x} = x = a^{\log_a x} = b^{(\log_b a)(\log_a x)} \]

\[ \log_b x = (\log_b a)(\log_a x) \]

\[ \log_a x = \log_b x / \log_b a \]

therefore \( \log_a x = O(\log_b x) \) for any \( a \) and \( b \)
\[ y^{\log x} = x^{\log y} \]

\[ x^{\log_b y} = \]

\[ y^{\log_y (x \log_b y)} = \]

\[ y^{(\log_b x / \log_b y) \log_b y} = \]

\[ y^{\log_b x} \]
Linear Time: $O(n)$

**Linear time.** Running time is proportional to the size of the input.

**Computing the maximum.** Compute maximum of $n$ numbers $a_1, \ldots, a_n$.

```text
max ← $a_1$
for $i = 2$ to $n$ {
  if ($a_i > max$)
    max ← $a_i$
}
```

Also $\Theta(n)$?
**Merge.** Combine two sorted lists \( A = a_1, a_2, \ldots, a_n \) with \( B = b_1, b_2, \ldots, b_n \) into a single sorted list.

\[
i = 1, j = 1
\]

**while** (both lists are nonempty) {

  **if** \( a_i \leq b_j \) append \( a_i \) to output list and increment \( i \)

  **else** append \( b_j \) to output list and increment \( j \)

} append remainder of nonempty list to output list

**Claim.** Merging two lists of size \( n \) takes \( O(n) \) time.
Polynomial evaluation. Given

\[ A(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (a_n \neq 0) \]

Evaluate \( A(x) \)

How not to do it:

\[ a_n \exp(x,n) + a_{n-1} \exp(x,n-1) + \ldots + a_1 x + a_0 \]

Why not?
How to do it: Horner's rule

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0 = \]
\[ (a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_1) x + a_0 = \ldots = \]
\[ (\ldots (a_n x + a_{n-1}) x + a_{n-2}) x \ldots + a_1) x + a_0 \]

\[ y = a[n] \]
\[ \text{for } (i=n-1; i>=0; i--) \]
\[ y = y * x + a[i] \]
Polynomial evaluation using Horner: complexity

Lower bound: $\Omega(n)$ because we need to access each $a[i]$ at least once

Upper bound: $O(n)$

Closed problem!

But what if $A(x) = x^n$
A(x) = x^n

Recurrence:

\[ x^{2n} = x^n \times x^n \quad x^{2n+1} = x \times x^{2n} \]

def pwr(x, n):
    if (n==0):
        return 1
    if odd(n):
        return x * pwr(x, n-1)
    else:
        a = pwr(x, n/2)
        return a * a

Complexity?
A glass-dropping experiment

- You are testing a model of glass jars, and want to know from what height you can drop a jar without its breaking. You can drop the jar from heights of 1,...,n foot heights. Higher means faster means more likely to break.
- You want to minimize the amount of work (number of heights you drop a jar from). Your strategy would depend on the number of jars you have available.
  - If you have a single jar:
    - do linear search (\(O(n)\) work).
  - If you have an unlimited number of jars:
    - do binary search (\(O(\log n)\) work)
  - Can you design a strategy for the case you have 2 jars, resulting in a bound that is strictly less than \(O(n)\)?

http://xkcd.com/510/
$O(n \log n)$ Time

Often arises in divide-and-conquer algorithms like mergesort.

```python
def mergesort(A):
    if len(A) <= 1:
        return A
    else:
        return merge(mergesort(left half(A)), mergesort(right half(A)))
```
Merge Sort - Divide

\{7,3,2,9,1,6,4,5\}

\{7,3,2,9\} \quad \{1,6,4,5\}

\{7,3\} \quad \{2,9\}

\{7\} \quad \{3\} \quad \{2\} \quad \{9\}

\{1,6\} \quad \{4,5\}

\{1\} \quad \{6\} \quad \{4\} \quad \{5\}
Merge Sort - Merge

{1,2,3,4,5,6,7,9}

{2,3,7,9}  {1,4,5,6}

{3,7}  {2,9}  {1,6}  {4,5}

{7}  {3}  {2}  {9}  {1}  {6}  {4}  {5}
mergesort(A):
    if len(A) <= 1 return A
    else return merge(mergesort(left half(A)), mergesort(right half(A)))
Quadratic Time: $O(n^2)$

Quadratic time example. Enumerate all pairs of elements.

Closest pair of points. Given a list of $n$ points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```plaintext
min ← (x_1 - x_2)^2 + (y_1 - y_2)^2
for i = 1 to n {
    for j = i+1 to n {
        d ← (x_i - x_j)^2 + (y_i - y_j)^2
        if (d < min)
            min ← d
    }
}
```

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.
Example 1: Matrix multiplication

Tight?

Example 2: Set disjoint-ness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, \ldots, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```plaintext
foreach set $S_i$ {
    foreach other set $S_j$ {
        foreach element $p$ of $S_i$ {
            determine whether $p$ also belongs to $S_j$
        }
        if (no element of $S_i$ belongs to $S_j$)
            report that $S_i$ and $S_j$ are disjoint
    }
}
```

what do we need for this to be $O(n^3)$?
Largest interval sum

Given an array $A[0],...,A[n-1]$, find indices $i,j$ such that the sum $A[i] +... +A[j]$ is maximized.

Naïve algorithm:

maximum_sum = - infinity
for $i$ in range($n-1$):
    for $j$ in range($i,n$):
        current_sum = $A[i] +... +A[j]$
        if current_sum >= maximum_sum:
            maximum_sum = current_sum
            save the values of $i$ and $j$

big $O$ bound?

Can we do better?

Example:

$A = [2, -3, 4, 2, 5, 7, -10, -8, 12]$
Polynomial Time: $O(n^k)$ Time

**Independent set of size** $k$. Given a graph, are there $k$ nodes such that no two are joined by an edge?

$k$ is a constant

**$O(n^k)$ solution.** Enumerate all subsets of $k$ nodes.

```
foreach subset S of k nodes {
    check whether S in an independent set
    if (S is an independent set)
        report S is an independent set
}
```

- Check whether $S$ is an independent set $= O(k^2)$.
- Number of $k$ element subsets $= \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$
  - poly-time for $k=17$, but not practical
Exponential Time

**Independent set.** Given a graph, what is the maximum size of an independent set?

**$O(n^2 2^n)$ solution.** Enumerate all subsets.

```plaintext
S* ← φ
foreach subset S of nodes {
    check whether S in an independent set
    if (S is largest independent set seen so far)
        update S* ← S
}
```

For some problems (e.g. TSP) we need to consider all permutations. The factorial function $(n!)$ grows much faster than $2^n$.
Questions

1. Is $2^n \ O(3^n)$ ?
2. Is $3^n \ O(2^n)$
3. Is $2^n \ O(n!)$ ?
4. Is $n! \ O(2^n)$
Some problems (such as matrix multiply) have a polynomial complexity solution: an $O(n^p)$ time algorithm solving them. (p constant)

Some problems (such as Hanoi) take an exponential time to solve: $\Theta(p^n)$ (p constant)

For some problems we only have an exponential solution, but we don't know if there exists a polynomial solution. Trial and error algorithms are the only ones we have so far to find an exact solution. If we would always make the right guess these algorithms would take polynomial time. Therefore we call these problems NP (we will discuss NP later)
Some NP problems

**TSP**: Travelling Salesman
- given cities $c_1, c_2, \ldots, c_n$ and distances between all of these, find a minimal tour connecting all cities.

**SAT**: Satisfiability
- given a boolean expression $E$ with boolean variables $x_1, x_2, \ldots, x_n$ determine a truth assignment to all $x_i$ making $E$ true.
Back tracking searches (walks) a state space, at each choice point it guesses a choice.

In a leaf (no further choices) if solution found OK, else go back to last choice point and pick another move.

NP is the class of problems for which we can check in polynomial time whether it is correct (certificates, later)
Coping with intractability

NP problems become intractable quickly
TSP for 100 cities?

How would you enumerate all possible tours? How many?

Coping with intractability:
- Approximation: Find a nearly optimal tour
- Randomization: use a probabilistic algorithm using "coin tosses" (eg prime witnesses)