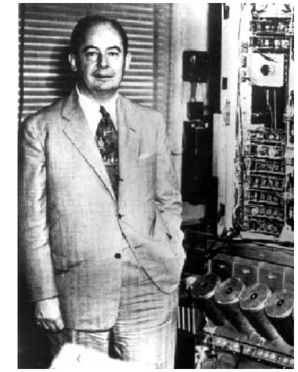
Divide and Conquer

Recurrence Relations

Divide-and-Conquer

Strategy:
- Break up problem into parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

MergeSort



Jon von Neumann (1945)

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

A	L	G	O	R	I	T	H	M	S

A	L	G	O	R	I	T	H	M	S	divide	O(1)
A	G	L	O	R	H	I	M	S	T	sort	2T(n/2)

A	G	H	I	L	M	O	R	S	T	merge	O(n)

Complexity of merge

time

$O(n)$

space

$O(n)$

Can you do it in less than 2n?

A Recurrence Relation for MergeSort

T(n) = number of comparisons required to mergesort an input of size n.

Mergesort recurrence.

$$T(n) \le \begin{cases} c & \text{if } n = 1 \\ \underbrace{T\left(\lceil n/2 \rceil\right)}_{\text{solve left half}} + \underbrace{T\left(\lfloor n/2 \rfloor\right)}_{\text{solve right half}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$

Recurrence Relations

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one of more of the previous terms of the sequence, namely, $a_0, a_1, \ldots a_{n-1}$, for all integers n with $n \geq n_0$ where n_0 is a nonnegative integer.

A sequence is defined by a recurrence relation + initial conditions ("base cases")

Example: Towers of Hanoi:

$$a_n = 2a_{n-1} + 1,\ a_1 = 1$$

A Recurrence Relation for MergeSort

T(n) = number of comparisons required to mergesort an input of size n.

Mergesort recurrence.

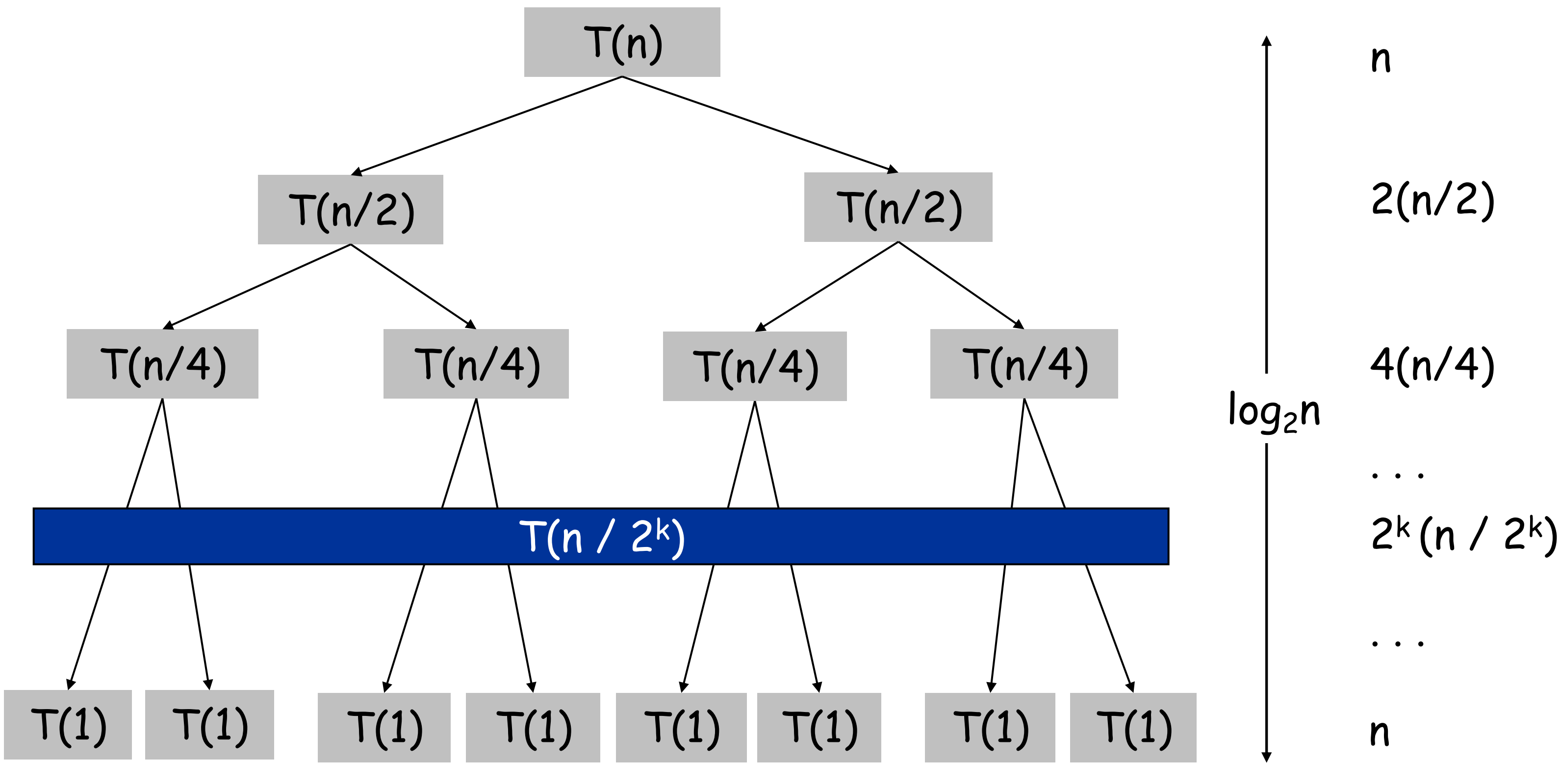
$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{T(n/2)}_{\text{solve left half}} + \underbrace{T(n/2)}_{\text{solve right half}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. We assume n is a power of 2 and replace $\leq$ with = (we only care about the order of magnitude)

Unrolling the recursion

$$\mathrm{T}(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$



$n/2^k = 1$ **when** $k = \log_2 n$

$n \log_2 n$

Repeated substitution

Claim. If T(n) satisfies this recurrence, then T(n) = cn $\log_2$ n.

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$

For n > 1:

$$\begin{aligned} T(n) &= 2T(n/2) + cn \\ &= 4T(n/4) + cn + 2n/2 \\ &= 8T(n/8) + cn + cn + 4cn/4 \\ &\cdots \\ &= 2^{\log_2 n} T(1) + \underbrace{cn + \cdots + cn}_{\log_2 n} \\ &= O(n \log_2 n) \end{aligned}$$

This reaches T(1) when
n = $2^{\log_2 n}$
by definition of $\log_2$n

Towers of Hanoi

Example: Towers of Hanoi, move all disks to third peg without ever placing a larger disk on a smaller one.

What's the recurrence relation?

Let's solve it by repeated substitution:

- Unroll the recurrence
- Identify a pattern
- Determine how often the pattern occurs before base case is hit, and sum over all the levels of the recursion

Hanoi by repeated substitution

$f_1=1$

$f_n= 2\, f_{n-1}+1 = 2(2f_{n-2}+1)+1 = 4f_{n-2}+2+1 = 4(2f_{n-3}+1)+2+1=$

$= 8f_{n-3}+4+2+1$

$= 2^3 f_{n-3}+ \sum_{i=0}^{2} 2^i \quad = 2^4 f_{n-4}+ \sum_{i=0}^{3} 2^i \quad = 2^k f_{n-k}+ \sum_{i=0}^{k-1} 2^i$

After n-1 substitutions, $k = n-1$, $\quad f_{n-(n-1)}= f_1=1$,and then

$2^{n-1} f_1+ \sum_{i=0}^{n-2} 2^i \quad = 2^{n-1}+ \sum_{i=0}^{n-2} 2^i$ $= \mathbf{2^n - 1 = O(2^n)}$

Repeated substitution for Binary Search

What's the recurrence relation for binary search?

Apply repeated substitution to solve it.

Finding maximum in unsorted array

Algorithm:

- If n=1, then element is the max.
- If n>1, divide array in half, find max of each and choose max of the two

Recurrence relation?

Solve by repeated substitution

$f(n) = 2f(n/2)+1 = 4f(n/4) + 2 + 1 = \ldots 2^k(f(n/2^k)) + 2^{k-1} + 2^{k-2}+\ldots+1 =$

$2.2^k-1 = 2n-1$

when $k = \log_2 n$ $n=2^k$ and $f(n/2^k) = f(1)=1$

STUDY YOUR LOGs (see Orders of magnitude lecture notes)
Useful trick: $y^{\log x} = x^{\log y}$
Also: $x^0+x^1+\ldots+x^n = (x^{n+1}-1)/(x-1)$ (geometric series)

The Master Theorem

Let f be an increasing function that satisfies

$$f(n) = a \cdot f(n/b) + c \cdot n^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer > 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) = \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

From section 7.3 in Rosen

mergesort: Recurrence Analysis

$$f(n) = a \cdot f(n/b) + cn^d$$

a =
b =
d =
O(?)

$$f(n) = \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$