NP and computational intractability

Kleinberg and Tardos, chapter 8

Major Transition

So far we have studied certain algorithmic patterns
- Greedy,
- Divide and conquer,
- Dynamic programming

to develop efficient algorithms.

Now we want to classify and quantify problems that cannot be solved efficiently.

Our tool for doing this is another algorithmic pattern: **reduction**

Reduction transforms the input and output of an algorithm so that it can be used to solve a different problem.

Reductions

An engineer is handed an **empty** kettle and is asked to make tea.
The engineer fills up the kettle, boils the water and makes tea.

A mathematician is handed a **full** kettle and is asked to make tea.

What do you think the mathematician does?

The mathematician empties the kettle
and hands it to the engineer ☺

Reductions

Reuse, Reduce, Recycle…

If we have a solution to one problem and we can use this solution to solve another problem, we do not need to write a new program, we can **reuse** the existing code, and **reduce** the new problem (change its input (and output)), so it can use the existing code to solve it.

eg 1: We have a **max heap**, but we need a **min heap**. How can we use the max heap to perform min heap operations, without changing one bit of the max heap code?

Insert(x): InsertMaxHeap(-x);

Extract(): x=ExtractMaxHeap(); return –x;

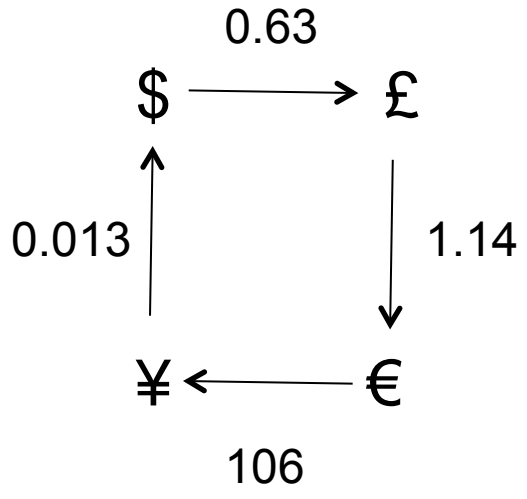
Arbitrage

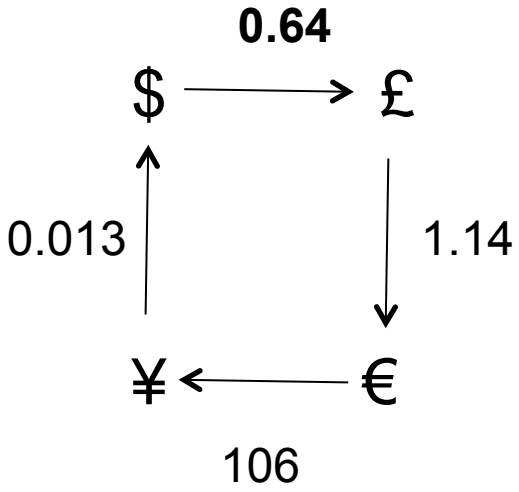
We have a large number of currencies and conversion rates from each currency to each other currency.

Is there a cyclic sequence of currency exchanges that provides a profit? eg:

loss: $ → £ → € → ¥ → $: 0.63*1.14*106*0.013 = 0.981

profit: $ → £ → € → ¥ → $: 0.**64***1.14*106*0.013 = 1.005





Arbitrage

We have a large number of currencies and conversion rates from each currency to each other currency.

Is there a cyclic sequence of currency exchanges that provides a profit?

Is there an algorithm that we studied, that we can use to solve his problem?

Bellman Ford: finding negative cycles in a graph

How would we reduce to that algorithm?

rate → -(log rate)

Try it for: 2 ¼ 4 ¼ and for: 2 ½ 4 ¼ and 2 1 4 ¼

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns	Example
Greedy.	$O(n \log n)$ interval scheduling
Divide-and-conquer.	$O(n \log n)$ closest point pair
Dynamic programming.	$O(n^2)$ sequence alignment
Reductions.	
Approximation algorithms.	
Randomized algorithms.	

Algorithm design anti-patterns	
NP-completeness.	$O(n^k)$ algorithm unlikely.
Undecidability.	No algorithm possible.

Classifying problems

Q. Which problems will we be able to solve in practice?

A working definition: P

Those with polynomial-time algorithms.

Some problems provably require exponential-time.

- Towers of Hanoi
- Generate all permutations of 1 … n

Grey area:

A huge number of fundamental problems have defied classification for decades.

We don't know of a polynomial algorithm for them, and we cannot prove that no polynomial algorithm exists.

Polynomial-Time Reductions

Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

Reduction. Problem Y polynomially reduces to problem X if arbitrary instances of problem Y can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to black box that solves problem X.
- Because polynomial (plus or times) polynomial is polynomial

Notation. $Y \leq_P X$.

Remarks.

- We pay for the time to write down instances sent to black box
  $\Rightarrow$ inputs of X must be of polynomial size.

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $Y \leq_P X$ and X can be solved in polynomial-time, then Y can also be solved in polynomial time.

Establish intractability. If $Y \leq_P X$ and Y cannot be solved in polynomial-time, then X cannot be solved in polynomial time.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$ they are as hard, notation: $X \equiv_P Y$.

Optimization vs Decision Problems

Decision problem

Does there exist an independent set of size ≥ k?

Optimization problem

Find independent set of **maximum** cardinality.

Easier to focus on **decision problems** (yes / no answers)

This is what we will do.

Reduction Strategies

- Reduction by equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

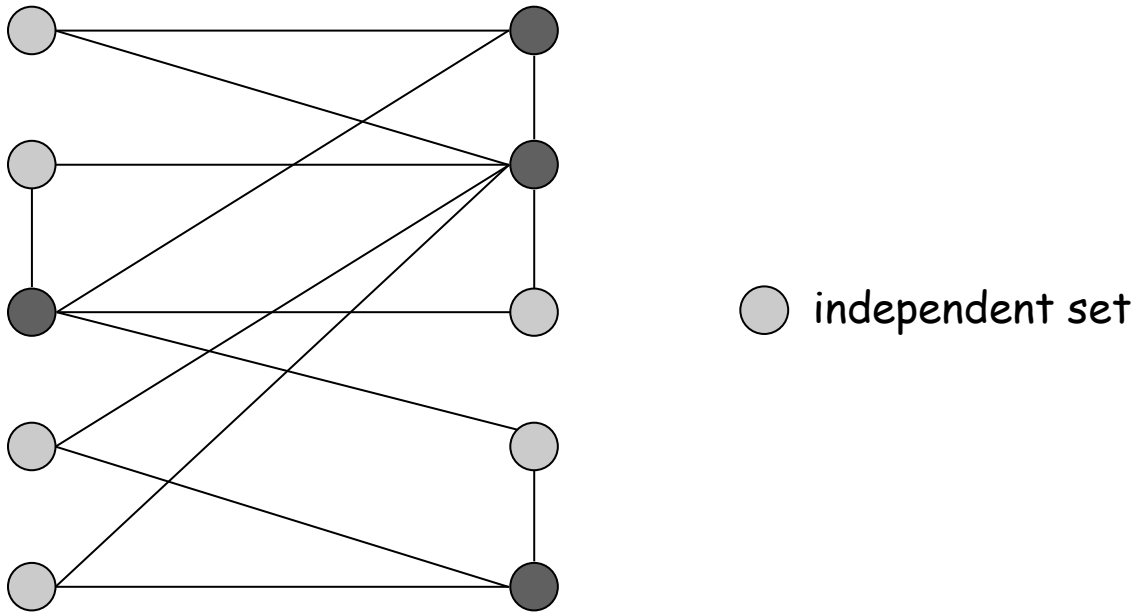
Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge **at most one** of its endpoints is in S?

(ie no edges join nodes in S)

Is there an independent set of size ≥ 6? Yes.

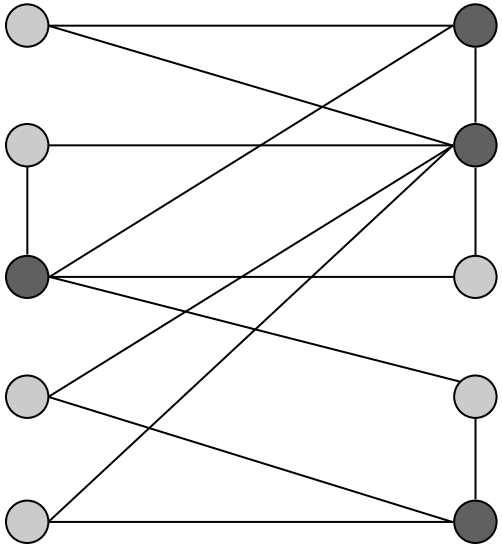
Is there an independent set of size ≥ 7? No.



Vertex Cover

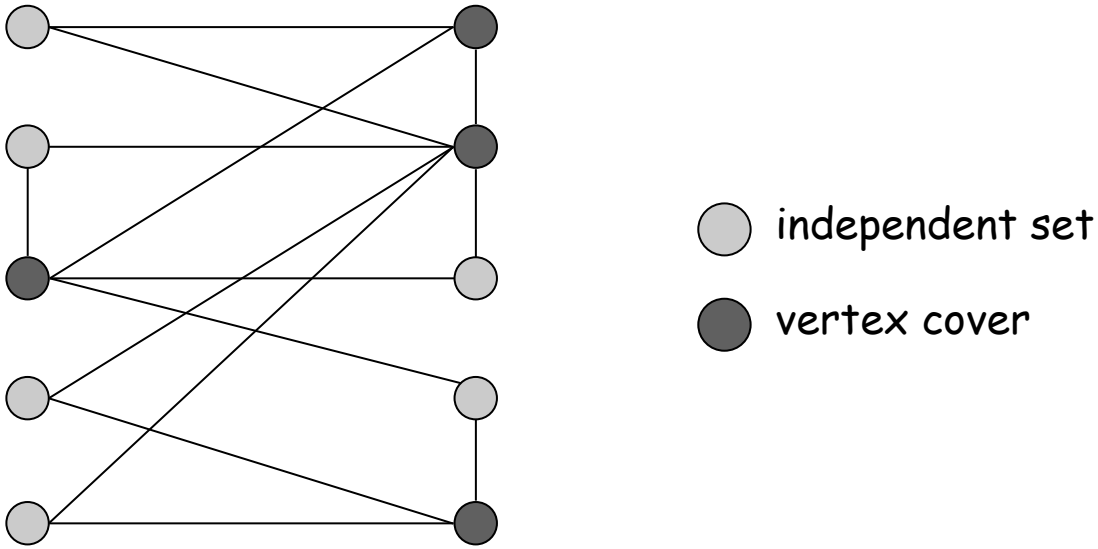
VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, **at least one** of its endpoints is in S?
(ie, the nodes in S cover all edges in E)

Is there a vertex cover of size ≤ 4? Yes.
Is there a vertex cover of size ≤ 3? No.



Vertex Cover and Independent Set

Claim. Let G=(V, E) be a graph. Then S is an independent set iff V–S is a vertex cover.



Vertex Cover and Independent Set

Claim. Let $G=(V, E)$ be a graph. Then S is an independent set iff $V - S$ is a vertex cover.

$\Rightarrow$

- Let S be an independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.
- Thus, $V - S$ covers (u, v).

$\Leftarrow$

- Let $V - S$ be a vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since $V - S$ is a vertex cover, and therefore all edges have at least one node in $V - S$.
- Thus, no two nodes in S are joined by an edge $\Rightarrow S$ independent set. ▪

Vertex Cover and Independent Set

S is an independent set iff V – S is a vertex cover.

This shows:

VERTEX-COVER $\leq_p$ INDEPENDENT-SET and
INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

from which we conclude:

VERTEX-COVER $\equiv_p$ INDEPENDENT-SET.

Reduction from Special Case to General Case

In this case we do not prove $X \equiv_P Y$, we prove $X \leq_P Y$

Eg: Vertex cover versus set cover

Vertex cover is phrased in terms of graphs

Set cover is phrased, more generally, in terms of sets

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, is there a collection of at most k of these sets whose union is equal to U?

Example:

U = { 1, 2, 3, 4, 5, 6, 7 }

k = 2

S_1 = {3, 7}

S_2 = {3, 4, 5, 6}

S_3 = {1}

S_4 = {2, 4}

S_5 = {5}

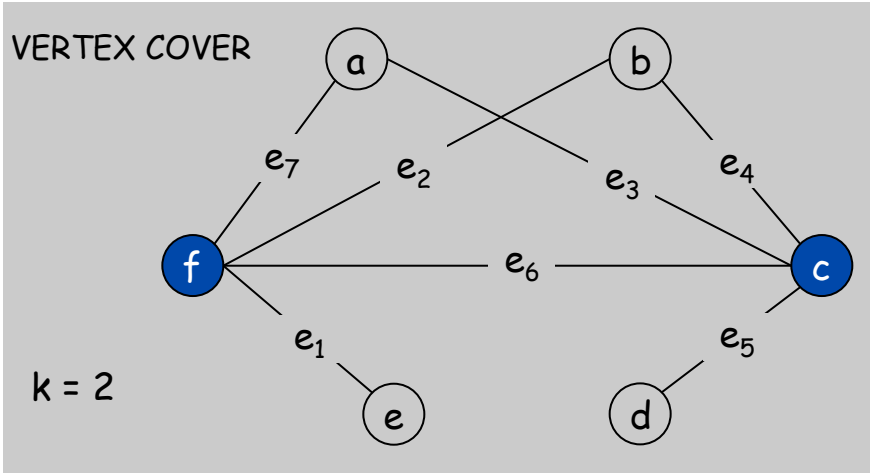
S_6 = {1, 2, 6, 7}

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Proof. Given a VERTEX-COVER instance G = (V, E), k, we reduce it to a set cover instance with U=E and a subset S_v for each v in V

- Create SET-COVER instance:
  - $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size $\leq k$ iff vertex cover of size $\leq k$.
- Hence we have shown that VERTEX-COVER $\leq_P$ SET-COVER ▪



SET COVER

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2

$S_a = \{3, 7\}$
$S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$
$S_d = \{5\}$
$S_e = \{1\}$
$S_f = \{1, 2, 6, 7\}$

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$ $(\overline{x_i} = not\ x_i)$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF):
A propositional formula Φ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given a CNF Φ, does it have a satisfying truth assignment?
3-SAT: SAT where each clause contains exactly 3 literals.

Example: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: x_1 = true, x_2 = true x_3 = false.

What is a necessary and sufficient condition for a CNF to be true?

Each clause must have one literal evaluating to true

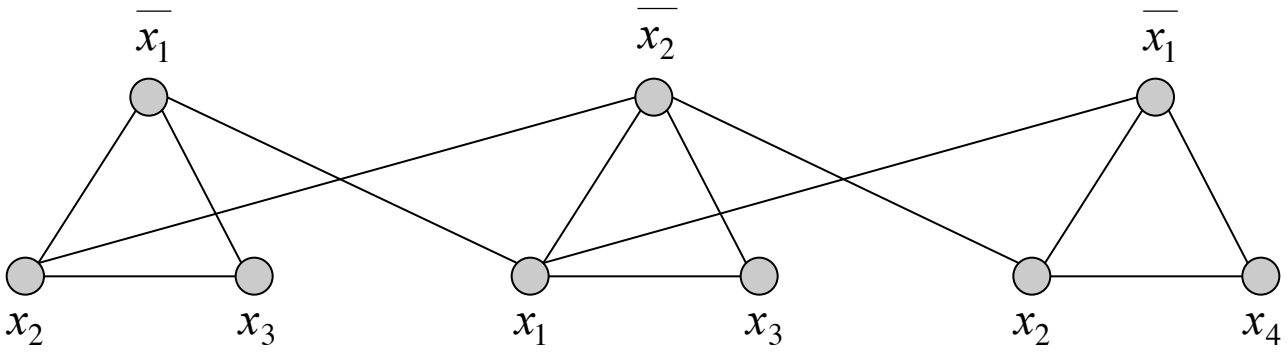
3-SAT Reduces to Independent Set

Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Proof. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.
- We call these connected triangles **gadgets**, creating a useful instance of independent set.



$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3 \right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3 \right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4 \right)$$

3-SAT Reduces to Independent Set

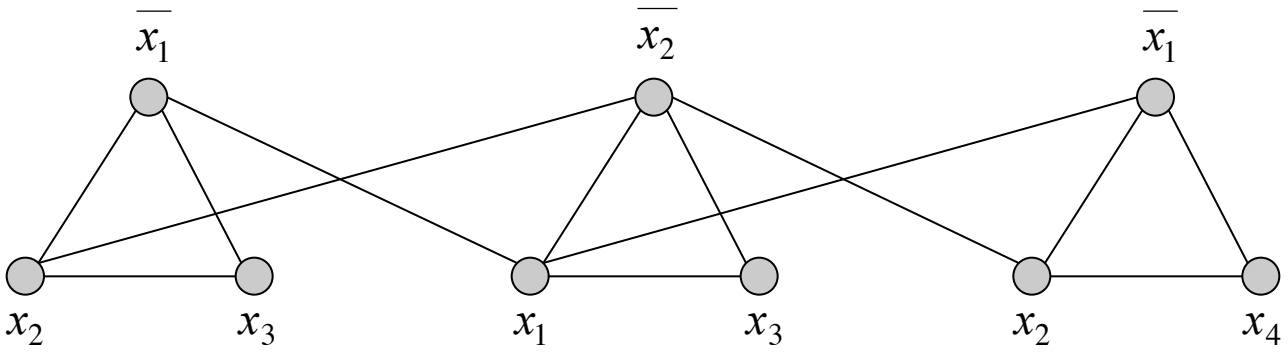
Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

number of clauses, ie triangles

Proof. $\Rightarrow$ Let S be independent set of size k.

1) S must contain exactly one vertex in each triangle, and cannot contain both x_i and $\overline{x_i}$ (they are connected)
2) Set these literals to true, there are no conflicts, because (1)
3) All clauses are satisfied.

$\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

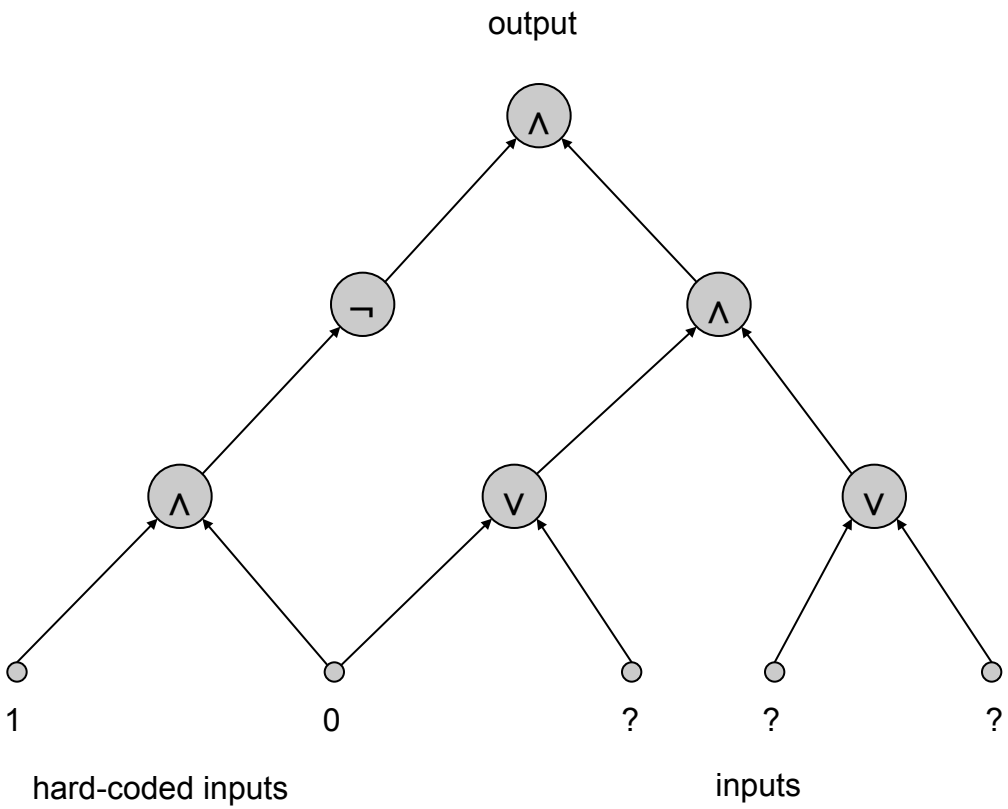


$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Circuit Satisfiability

CIRCUIT-SAT. A combinational circuit is a directed acyclic graph built out of AND, OR, and NOT nodes. Given such circuit, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1



CIRCUIT-SAT $\leq_p$ 3-SAT

- Let K be any circuit.
- Create a 3-SAT variable x_i for each circuit element i.
- Make 3-SAT clauses compute values for each circuit-SAT node, eg.:

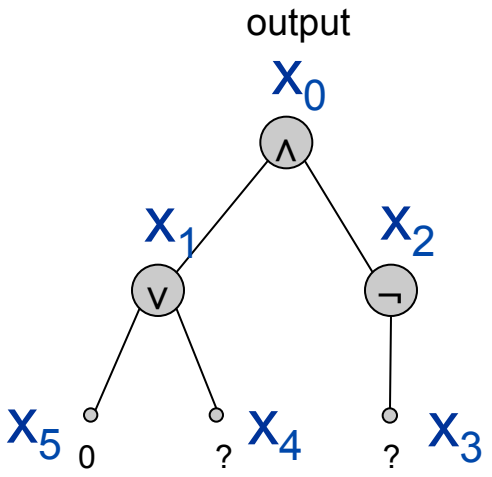
  $x_2 = \neg\, x_3 \quad \Rightarrow$ add 2 clauses: $\quad x_2 \vee \overline{x_3}\,,\ \overline{x_2} \vee \overline{x_3}$

  $x_1 = x_4 \vee x_5 \quad \Rightarrow$ add 3 clauses: $\quad x_1 \vee \overline{x_4}\,,\ x_1 \vee \overline{x_5}\,,\ \overline{x_1} \vee x_4 \vee x_5$

  $x_0 = x_1 \wedge x_2 \quad \Rightarrow$ add 3 clauses: $\quad \overline{x_0} \vee x_1\,,\ \overline{x_0} \vee x_2\,,\ x_0 \vee \overline{x_1} \vee \overline{x_2}$

- Hard-coded input values and output value.
  - $x_5 = 0 \Rightarrow$ add 1 clause: $\overline{x_5}$
  - $x_0 = 1 \Rightarrow$ add 1 clause: x_0

- Final step: turn clauses of length < 3 into clauses of length exactly 3. **HOW?**



Independent Set $\leq_p$ CircSAT

To show that any Independent Set_k problem $\leq_p$ CircSAT we construct a circuit C with inputs and one output:

inputs:

1: **graph description**, in terms of all its edges
(1 bit per node pair)

2: **set description**, in terms of the nodes in the set
(1 bit per node)

output: one output bit: yes/no

The components of the circuit:

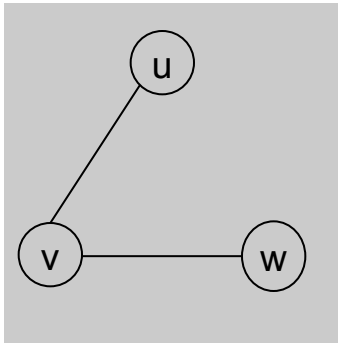
1: define the graph in terms of the edges it contains

2: check whether the set has a pair of nodes that is an edge in the graph.
If here is such a pair, the set is not independent

3: count whether there are at least k nodes in the set

Independent Set $\leq_p$ CircSAT

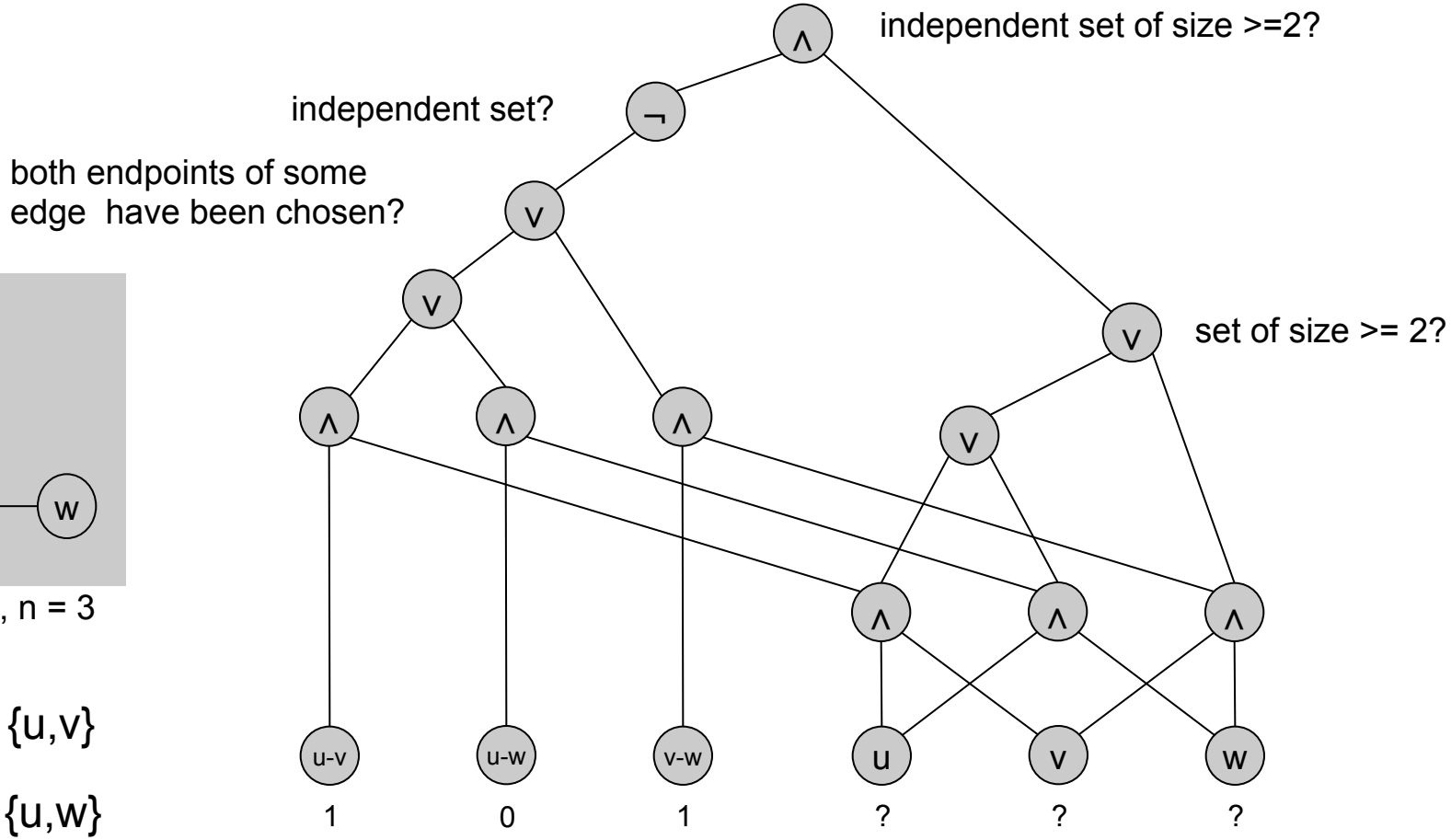
Example. Construction below creates a circuit C whose inputs can be set so that C outputs true iff graph G has an independent set of size >=2.



G = (V, E), n = 3

Try it for {u,v}

Try it for {u,w}



Review

Basic reduction strategies.

- Equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
Proof idea. Compose the two algorithms.

Example:

3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.