Assignment 3 – Logic and Reasoning

Print this sheet and fill in your answers. Please staple the sheets together. Turn in at the beginning of class on Friday, September 8.

Recall this about logic: Suppose there is no life on Mars. Then the statement: “All Martians are green” is a true statement. Any statement in logic is true if no counterexample can exist. If there is no example of it either, it is what is called vacuously true.

Hint: Several of these questions relate to the logic truth table for implication.

1. Suppose you know that A ⇒ B is true and that A is true. What do these facts tell you about the value of B, if anything?

2. Suppose you know that A ⇒ B is true and that A is false. What do these facts tell you about the value of B, if anything?

3. Suppose you know that A ⇒ B is true and B is false. What do these facts tell you about the value of A, if anything?

4. Tell whether the following implication statement is true under each of the following scenarios, or state that it can’t be determined: (All politicians are dishonest) ⇒ (The country is in trouble)
   a. All politicians are dishonest is a true statement.
   b. There exists an honest politician is a true statement
   c. The country is in trouble is a true statement.

5. Suppose: (All politicians are dishonest) ⇒ (The country is in trouble) is a true statement, but the country is in trouble is a false statement. Precisely what do these facts imply about politicians?

6. For which integer values of n is the following statement true? (n = 6) ⇒ (n + 1 = 7)
7. Use a truth table to solve for the values of A, B, given the following statements are true:

\[ \neg A \Rightarrow B, \neg B \Rightarrow \neg C, C \]

8. Here is a “proof” that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + 1 \) for all \( n \geq 1 \). For \( k \geq 1 \), let \( A_k \Rightarrow A_{k+1} \)

If \( A_k \) is false, the implication is automatically true. If \( A_k \) is true, then:

\[ \sum_{i=1}^{k} i = \frac{k(k+1)}{2} + 1, \quad \text{and} \quad \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) = \frac{k(k+1)}{2} + 1 + (k + 1) = \frac{(k+1)(k+2)}{2} + 1 \]

\[ \sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2} + 1 \] means \( A_{k+1} \) is true. \( A_k \Rightarrow A_{k+1} \) is true in this case, as well.

For each of the following statements, tell whether we have proved the statement true:

a. \( A_{100} \Rightarrow A_{101} \)

b. \( A_1 \Rightarrow A_2 \)

c. \( A_5 \) is true

d. \( A_5 \) is false

9. Joe Bloggs wants to show that \( A_k \) is true for arbitrary \( k \geq 1 \). He gives a correct proof that \( A_k \Rightarrow A_{k+1} \) for all \( k \geq 1 \), but he forgot the base case. Another student gives a correct proof that \( A_0 \) is false.

For each of the following tell whether we can conclude that it is true or false, or else indicate that there is not enough information.

a. \( A_9 \Rightarrow A_{10} \)

b. \( A_1 \Rightarrow A_2 \)

c. \( A_8 \)

d. \( A_1 \)

e. \( A_{10} \)

f. \( A_k \) is false for an infinite set of values \( k \)
10. Here is a proof by induction that all babies have the same eye color. The babies in the world are a set of babies, so it suffices to show that every set of babies has a uniform eye color.

Let $A_k$ be true for all $k \geq 1$, then this will prove that $A_n$ is true, where $n$ is the number of babies in the world. That will mean that all babies have the same eye color.

**Base Case:** $A_1$ is true because in every set of babies that has just one baby, there are no two babies in the set that have different eye colors.

**Induction step:** Let $k \geq 1$. If $A_k$ is false, then $A_k \Rightarrow A_{k+1}$ is true.

So suppose $A_k$ is true. Let $S = \{b_1, b_2, \ldots, b_k, b_{k+1}\}$ be an arbitrary set of $k+1$ babies. Then \{\{b_1, b_2, \ldots, b_k\}\} and \{\{b_2, b_3, \ldots, b_{k+1}\}\} are two sets of $k$ babies. Since $A_k$ is true all babies in each of these sets have the same eye color. Therefore, $b_1$ and $b_{k+1}$ have the same eye color as the babies in \{\{b_2, b_3, \ldots, b_{k+1}\}\}, so they have the same eye color as each other, and all babies in $S$ have the same eye color. Since $S$ is an arbitrary set of $k + 1$ babies, the same argument applies to every set of $k + 1$ babies, and $A_{k+1}$ is true: $A_k \Rightarrow A_{k+1}$ is true in this case also.

We conclude that whether or not $A_k$ is true, $A_k \Rightarrow A_{k+1}$ is true.

Which of the following have been successfully proved to be true by the arguments?

a. $A_5 \Rightarrow A_6$

b. $A_1$

c. $A_1 \Rightarrow A_2$

11. Suppose that in every set of two babies, both babies have the same eye color. What would this imply about the eye colors of all babies?
12. Here are some statements related to the book’s proof of correctness of Gale-Shapley.

- A: The set returned by every execution G-S is a stable matching.
- B: The set returned by an execution of G-S is never a stable matching.
- C: A set S returned by some execution of G-S is not a stable matching.
- D: In S, there exist pairs (m,w) and (m',w') such that m prefers w' to w and w' prefers m to m'.
- E: In S, there exist pairs (m,w) and (m', w') such that m prefers w' to w and w prefers m' to m.
- F: w' rejects m, and traded up to higher and higher men in her list, culminating in m'.

Which of the following best reflects the logical structure of the proof?

a. Using logic, it shows that $C \Rightarrow D, D \Rightarrow F$ and $F \Rightarrow ! (D)$ are true statements, without yet resolving the values of $C, D, and F$. $(D \Rightarrow F) AND (F \Rightarrow ! (D)) \Rightarrow ! (D)$ is true. Therefore $Not(D)$ is true. Since $C \Rightarrow D, ! (D) \Rightarrow Not(C)$, so $! (C)$ is true. $C \Rightarrow ! (A)$, A is true.

b. Using logic, it shows that $B \Rightarrow D, D \Rightarrow F$ and $F \Rightarrow ! (D)$ are true statements, without yet resolving the values of $B, D, and F$. $(D \Rightarrow F) AND (F \Rightarrow ! (D)) \Rightarrow ! (D)$ is true. $B = ! (A)$, so A is true.

c. Using logic, it shows that $C \Rightarrow D, D \Rightarrow E$, and $E \Rightarrow ! (D)$ are true statements, without yet resolving the values of $C, D, and E$. $(D \Rightarrow E) AND (E \Rightarrow ! (D)) \Rightarrow ! (D)$ is true. Therefore $! (D)$ is true. Since $C \Rightarrow D, ! (D) \Rightarrow ! (C)$, so $! (C)$ is true. $C = ! (A)$, so A is true.
13. In the stable matching problem, let two lovebirds be a man and a woman who each first on the other’s list. Let a last resort denote a man and a woman that are each last on the other’s list. For each of the following, either prove that the statement is true or prove that it is false. To prove a statement is false, come up with a counterexample. (For such a proof, it is desirable to find the smallest counterexample you can think of.) IF you want to prove the statement true, use proof by contradiction: assume that it is false and derive a contradiction.
   a. Claim: No stable matching contains a last resort.
   
   b. Claim: Every stable matching contains a pair of lovebirds. A pair of lovebirds is two people, not two couples.
   
   c. Claim: Whenever there is a pair of lovebirds. They are married in every stable matching.

14. We have shown that the number of proposals until the algorithm halts is at most \( n^2 \), since none of the \( n \) men can make more than \( n \) proposals. We haven’t proved that this many proposals could ever happen. Show that it can be the case that there are at least \( \frac{n^2}{2} \) proposals before it halts.

15. Prove or disprove: \( n^2 + n + 17 \) is prime for all \( n \geq 0 \).