Convex Hull
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Convex Hull Problem

Smallest convex polygon P for which each point in a set Q of points is either on the boundary of P or in its interior.

Graham’s scan: O(n \lg n)
Jarvis’s march: O(nh)

Can also use a divide-and-conquer approach and divide the points into a left and right group, ... O(n \lg n) if combining answers is done in O(n)
Convex Hull Applications

2-dimensional farthest-pair problem – must be on the convex hull
Graham’s Scan

• Stack of candidate points
• Pop points that are not vertices
• Result is convex hull vertices CH(Q)
  • counterclockwise order
**Graham-Scan**\( (Q)\)

1. let \( p_0 \) be the point in \( Q \) with the minimum \( y \)-coordinate, 
or the leftmost such point in case of a tie
2. let \( (p_1, p_2, \ldots, p_m) \) be the remaining points in \( Q \), 
sorted by polar angle in counterclockwise order around \( p_0 \) 
   (if more than one point has the same angle, remove all but 
   the one that is farthest from \( p_0 \))
3. let \( S \) be an empty stack
4. \textbf{Push}(p_0, S)
5. \textbf{Push}(p_1, S)
6. \textbf{Push}(p_2, S)
7. \textbf{for} \( i = 3 \) \textbf{to} \( m \)
8. \hspace{1em} \textbf{while} the angle formed by points \textbf{Next-To-Top}(S), \textbf{Top}(S), 
   and \( p_i \) makes a nonleft turn
9. \hspace{2em} \textbf{Pop}(S)
10. \hspace{2em} \textbf{Push}(p_i, S)
11. \textbf{return} \( S \)
Which Way?

\( \angle p_0 p_1 p_2 \) turns right or left?

Without computing angle?
Which Way?

∠p_0p_1p_2 turns right or left?

Use cross products: relation of p_0p_2 to p_0p_1

Compute cross product \( cp = (p_2 - p_0) \times (p_1 - p_0) \)

\( cp < 0: \) p_0p_2 counterclockwise to p_0p_1 - LEFT at p_1
\( cp > 0: \) p_0p_2 clockwise to p_0p_1 - RIGHT at p_1
\( cp = 0: \) p_0p_2p_1 are colinear
Complexity?

**Graham-Scan**($Q$)

1. let $p_0$ be the point in $Q$ with the minimum $y$-coordinate, or the leftmost such point in case of a tie
2. let $(p_1, p_2, \ldots, p_m)$ be the remaining points in $Q$, sorted by polar angle in counterclockwise order around $p_0$ (if more than one point has the same angle, remove all but the one that is farthest from $p_0$)
3. let $S$ be an empty stack
4. **PUSH**($p_0, S$)
5. **PUSH**($p_1, S$)
6. **PUSH**($p_2, S$)
7. for $i = 3$ to $m$
   8. while the angle formed by points **NEXT-TO-TOP**(S), **TOP**(S), and $p_i$ makes a nonleft turn
   9. **POP**($S$)
10. **PUSH**($p_i, S$)
11. return $S$
\[ Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \]
Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}
Jarvis’s March

Package wrapping: $O(nh)$, where $h$ is number of vertices in the convex hull

Tape the end of the paper to the lowest point in the set.
Pull to the right till taut, then up until it touches a point.
Continue around the set of vertices.
Jarvis’s March – more formally

Each next vertex has the smallest polar angle with respect to the previous vertex.

At the highest vertex we have all of the right chain.

From there we choose the vertex with the smallest polar angle, from the negative x-axis.

The left chain is complete when we get back to the 1st vertex.

Using 2 chains means we can compare angles without computing them like Graham’s.
Image Credits

3dconvex: [https://stackoverflow.com/questions/18416861/how-to-find-convex-hull-in-a-3-dimensional-space](https://stackoverflow.com/questions/18416861/how-to-find-convex-hull-in-a-3-dimensional-space)