Priority Queues/Heaps
CS 320, Fall 2017

Dr. Geri Georg, Instructor
georg@colostate.edu
You Just Bought a New Car
Have to Register It
Get a “now serving” Ticket
And Wait More....
What – Multiple Queues?
How about ONE priority queue?

How do we integrate multiple queues?
Have to decide priorities of old queues:
1. PWD
2. Military in uniform
3. Everybody else

PWD
Military
Everybody else

Highest priority
What’s a Heap?

A complete binary tree

Has the heap property:

all nodes have a lower value than the nodes in their subtrees.

We insert new nodes “at the end” to maintain a complete binary tree, then swap nodes as necessary to restore the heap property.
Add node to existing level

Start new level

insert 6
Implementation?
We can store a heap as an array if we can figure out an efficient indexing.

The root is at $A[1]$. For any node $i$, the left child is at index $2i$ and the right at $2i + 1$. 
insert 5
See Heap-Building in Action

Heap visualization -
http://www.cs.usfca.edu/~galles/visualization/Heap.html
Heap Complexity

How many nodes can we have?

Array is 0-31, size 32 = 2^5
but A[0] = ∞, so there is space for 31 nodes
Height of the tree?

If we think of the root as being at Height 0, how many nodes are there at each level? $2^h$, where $h$ is the height.
How many total nodes in the tree at any height?

- **root = Height 0, 1 node**
- **Height 1, 2 nodes**
- **Height 2**
- **$2^{h+1} - 1$ nodes**
Height of the tree for a given number of nodes?

- Root = Height = 0, 1 node
- Height = 1, 2 nodes
- H=3, n=15
- H=2, n=7
- H=4, n=31

Height = \( \log_2 n \)
Implementation

Whether we use arrays for the heap or not, we need functions to:

• Build the heap initially
• Extract the minimum element
• Insert an element
  • Add to heap & then either move it up or down to restore the heap property
  • Calls these latter Heapify-up and Heapify-down
Back to the animation:
1. What is the worst number of levels that a node will move up when it is added to the heap? (Remember it is always added to the last level, from the left side of the tree.)
   4 levels
2. What is the worst number of levels that a node will move down when the highest priority element is removed from the heap? (How does the animation fill in the “hole” created when the minimum is removed?)
   4 levels
Insert ‘5’ – watch the number of swaps and the array indices

...from index 19 to 9
Insert ‘5’ – watch the number of swaps and the array indices

...from index 9 to 4
Insert ‘5’ – watch the number of swaps and the array indices

...from index 4 to 2
Insert ‘5’ – watch the number of swaps and the array indices

...from index 2 to 1
Insert ‘5’ – watch the number of swaps and the array indices

Total movement was from index 19 to 1
What is the number of times the inserted was swapped in terms of the index, $i$?

Initially, $i = 19$, number of swaps was 4

$$\lfloor \log_2 19 \rfloor = 4 = \lfloor \log_2 n \rfloor \text{ since } n=19!$$

In the \textbf{worst} case, a node \textbf{at the end of the array}, at index $i = n$, will move up in the worst case $\log_2 n$ times in the array, which makes the complexity of Heapify-up $O(\log_2 n)$
How about downward movement? Worst case: root moves to the last position in the array, \( n \).

Assume it starts at \( i = 1 \), and ends at \( i = 19 \), how many swaps would this be in terms of \( i \)?

\[
\left\lfloor \log_2 19 \right\rfloor = 4, \text{ but this is the same as } \left\lfloor \log_2 n \right\rfloor
\]

In the **worst** case, the root will move down in \( \log_2 n \) times in the array, which makes the complexity of Heapify-down \( O(\log_2 n) \).
Implementation Complexity

• Extract the minimum element – $\Theta(1)$
• Heapify-up – $O(\log_2 n)$
• Heapify-down – $O(\log_2 n)$
• Insert
  • Add to the end of the array – $\Theta(1)$
  • Heapify-up/-down – $O(\log_2 n)$
• Build the heap initially – insert $n$ times, once for each element – $O(n\log_2 n)$
Image Credits

Priority Q/implementation types: https://www.cs.cmu.edu/~adamchik/15-121/lectures/Binary%20Heaps/heaps.html
dmv-bigwait: https://atypical60.com/2016/03/08/the-license-to-drive-me-crazy-a-day-at-the-dmv/
heapinsert: http://cs.lmu.edu/~ray/notes/pqueues/