Lecture 23: Hermite and Bezier Curves

November 16, 2017

Representing Curved Objects

- So far we've seen
  - Polygonal objects (triangles) and Spheres
- Now, polynomial curves
  - Hermite curves
  - Bezier curves
  - B-Splines
  - NURBS
- Bivariate polynomial surface patches

Beyond linear approximation

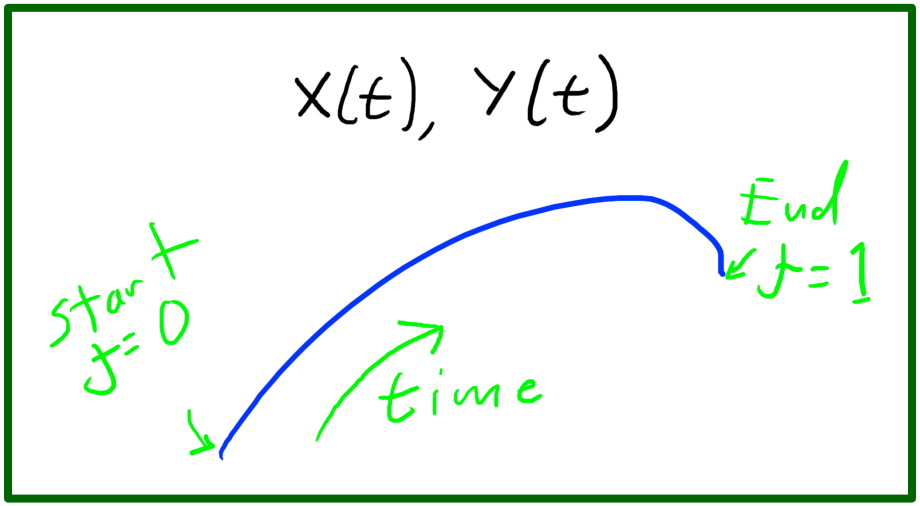
Instead of approximating everything by zillions of lines and planes, it is possible to approximate shapes using higher-order curves. Advantages:

- More compact
- Reduces "artifacts"

Use of sphere in ray tracer is an example of an implicit curve.

The Pen Metaphore

- Think of putting a pen to paper
- Pen position described by time t



Seeing the action of drawing is the key, so this static drawing only partly captures the point of this slide.

Design Criteria

- Local control of shape
- Smoothness and continuity
- Ability to evaluate derivatives
- Stability
- Ease of rendering

Review Forms - Explicit

- Explicit representation: y = f(x)

$$y = ax^3 + bx^2 + cx + d$$

- Drawbacks:
  - Multiple values of y for a single x impossible.
  - Not rotationally invariant.

Review Forms - Implicit

- Implicit representation: f(x, y, z) = 0

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

- Advantages:
  - On curve & relative distance to curve tests.
- Drawbacks:
  - Enumerating points on the curve is hard.
  - Extra constraints needed – half a circle?
  - Difficult to express and test tangents.

Parametric Representations

We will represent 3D curves using a parametric representation, introducing a new variable t:

$$Q(t) = \begin{vmatrix} x(t) & y(t) & z(t) \end{vmatrix}$$

Note that x, y and z are dependent on t alone, making it clear that there is only one free variable.

Think of t as time associated with movement along the curve.

Third Order Curves

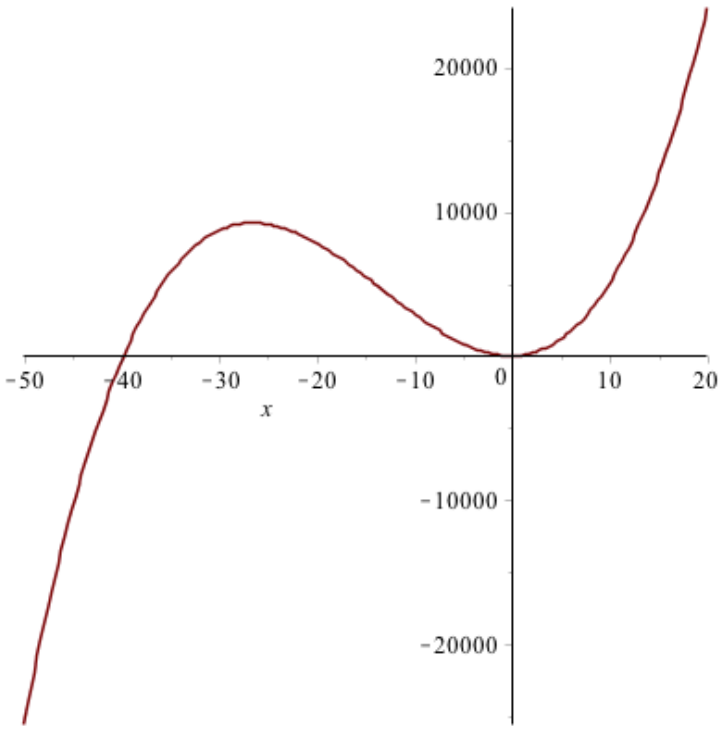
- Third-order functions are the standard:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

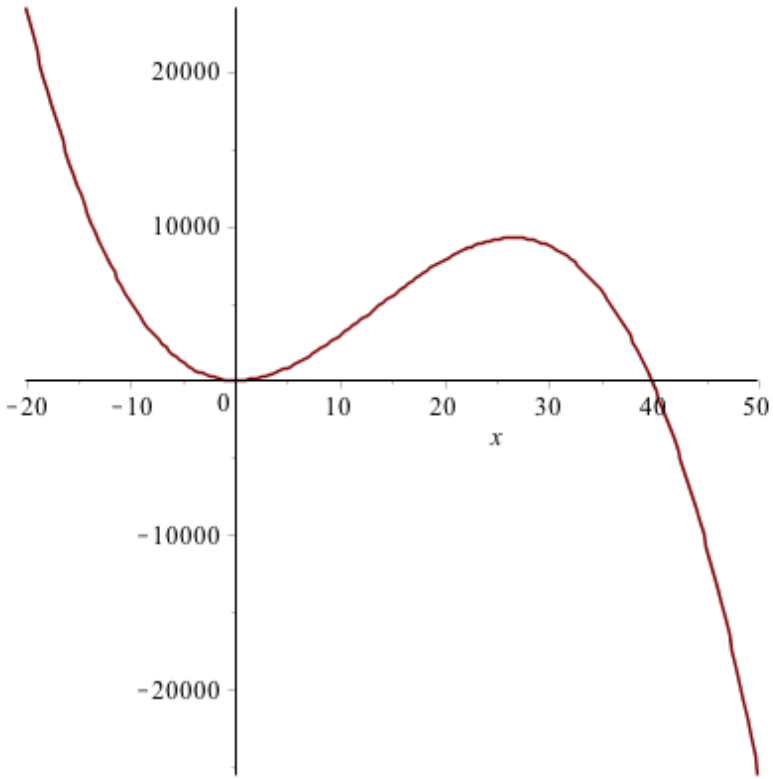
- Why 3?
  - Lower-order curves cannot be smoothly joined.
  - Higher-order curves introduce “wiggles”.
- Without loss of generality: 0 <= t <= 1.

Cubic Examples

$$x^3 + 40\,x^2 + 10\,x + 2$$



$$-x^3 + 40\,x^2 - 10\,x + 2$$



Notation

$$T = \left[t^3, t^2, t, 1 \right]$$

$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

$$Q(t) = T \cdot C$$

Alternatively: $Q(t)^T = C^T \cdot T^T$

Tangents to Cubic Curves

The derivative of Q(t) is its tangent:

$$\frac{d}{dt} Q(t) = [\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t)]$$

$$\frac{d}{dt} Q(t)_x = 3a_x t^2 + 2b_x t + c_x$$

$$\frac{d}{dt} Q(t) = [3t^2, 2t, 1, 0]\, C$$

The same matrix as on previous slide

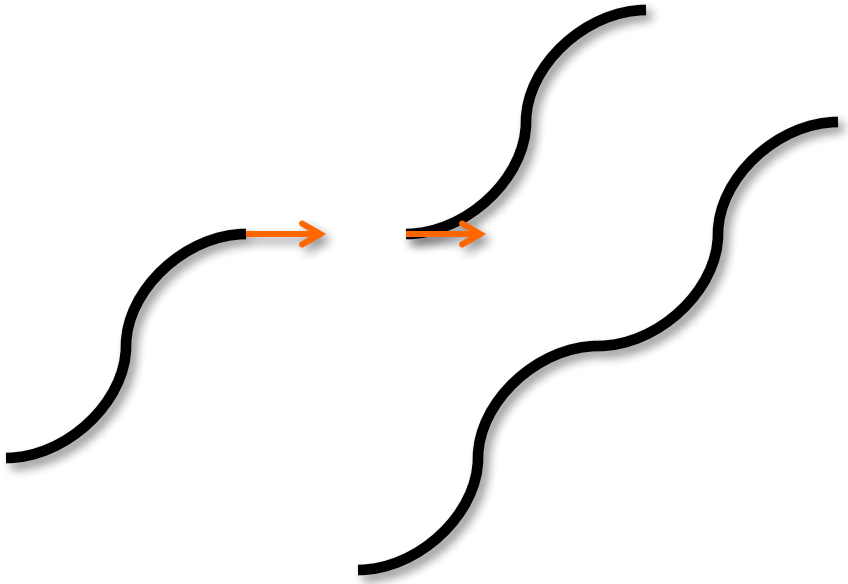
Again the time metaphor is useful, the tangent indicates instantaneous direction and speed.

Hermite Curves

We want curves that fit together smoothly.

To accomplish this, we would like to specify a curve by providing:

- The endpoints
- The 1st derivatives at the endpoints



The result is called a *Hermite Curve*.

Hermite Curves (cont.)

Since Q(t) = TC, we factor C into two matrices:

G (a 3x4 geometry matrix)

M (a 4x4 basis matrix)

such that C = G • M.

This step is a big deal. It makes thinking about curve geometry tractable.

Note: G will hold our geometric constraints (endpoints and derivatives), while M will be constant across all Hermite curves.

Let us concentrate on the x component:

$$P(t)_x = a_x t^3 + b_x t^2 + c_x t + d_x$$

Remember that its derivative is:

$$\frac{d}{dt} P(t)_x = 3a_x t^2 + 2b_x t + c_x$$

Therefore

$$\begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\, P(0)_x \\ d/dt\, P(1)_x \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix}$$

Therefore:

$$\begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix}^{-1} \begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\ P(0)_x \\ d/dt\ P(1)_x \end{vmatrix}$$

And taking the inverse:

$$\begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\ P(0)_x \\ d/dt\ P(1)_x \end{vmatrix}$$

The Hermite Matrix

OK, that was the x dimension.
How about the others?

They are, of course, the same:

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{vmatrix} = \underbrace{\begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}}_{\mathrm{M}} \begin{vmatrix} P(0) \\ P(1) \\ \dfrac{dP(0)}{dt} \\ \dfrac{dP(1)}{dt} \end{vmatrix}$$

Punchline

$$Q(t) = \begin{vmatrix} x(t) & y(t) & z(t) \end{vmatrix} = T \cdot C = T \cdot M_H \cdot \begin{bmatrix} P(0) \\ P(1) \\ d/dt\, P(0) \\ d/dt\, P(1) \end{bmatrix}$$

Since M_H and T are known, you can write down a cubic polynomial curve by inspection ending at points P(0) and P(1) with tangents d/dt P(0) and d/dt P(1).

Expand the Math – Look Inside

Recall Parametric Equation: $Q(t) = T\, M_H\, G$

Where

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x1 & y1 & z1 \\ x2 & y2 & z2 \\ dx1 & dy1 & dz1 \\ dx2 & dy2 & dz2 \end{bmatrix}$$

Fully Expanded (note transpose)

$$Q(t)^T = \begin{bmatrix} (2t^3 - 3t^2 + 1)\,x1 + (3t^2 - 2t^3)\,x2 + (-2t^2 + t^3 + t)\,dx1 + (-t^2 + t^3)\,dx2 \\ (2t^3 - 3t^2 + 1)\,y1 + (3t^2 - 2t^3)\,y2 + (-2t^2 + t^3 + t)\,dy1 + (-t^2 + t^3)\,dy2 \\ (2t^3 - 3t^2 + 1)\,z1 + (3t^2 - 2t^3)\,z2 + (-2t^2 + t^3 + t)\,dz1 + (-t^2 + t^3)\,dz2 \end{bmatrix}$$

If you Prefer

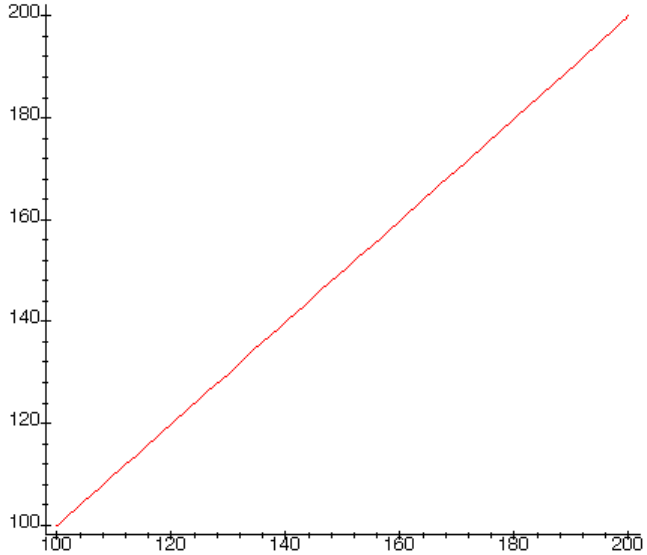
- There are two equivalent setups
- The difference is solely transposition

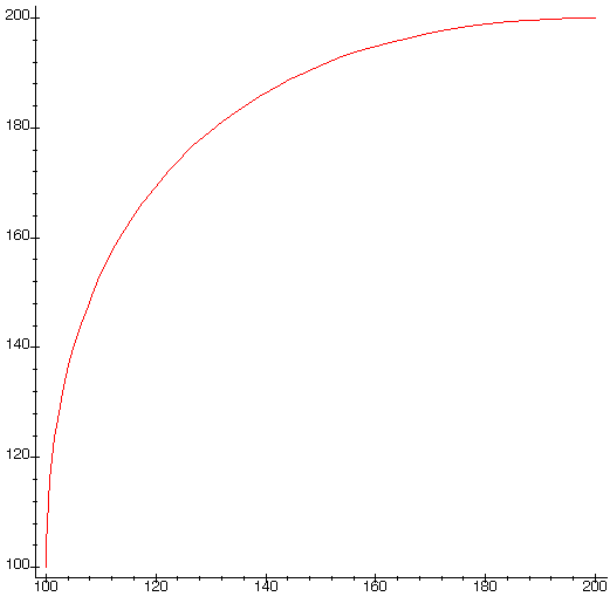
$$Q(t) = G\,M\,T$$

$$Q(t) = \begin{bmatrix} x1 & x2 & dx1 & dx2 \\ y1 & y2 & dy1 & dy2 \\ z1 & z2 & dz1 & dz2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} \left(2\,t^3 - 3\,t^2 + 1\right) x1 + \left(3\,t^2 - 2\,t^3\right) x2 + \left(-2\,t^2 + t^3 + t\right) dx1 + \left(-t^2 + t^3\right) dx2 \\ \left(2\,t^3 - 3\,t^2 + 1\right) y1 + \left(3\,t^2 - 2\,t^3\right) y2 + \left(-2\,t^2 + t^3 + t\right) dy1 + \left(-t^2 + t^3\right) dy2 \\ \left(2\,t^3 - 3\,t^2 + 1\right) z1 + \left(3\,t^2 - 2\,t^3\right) z2 + \left(-2\,t^2 + t^3 + t\right) dz1 + \left(-t^2 + t^3\right) dz2 \end{bmatrix}$$

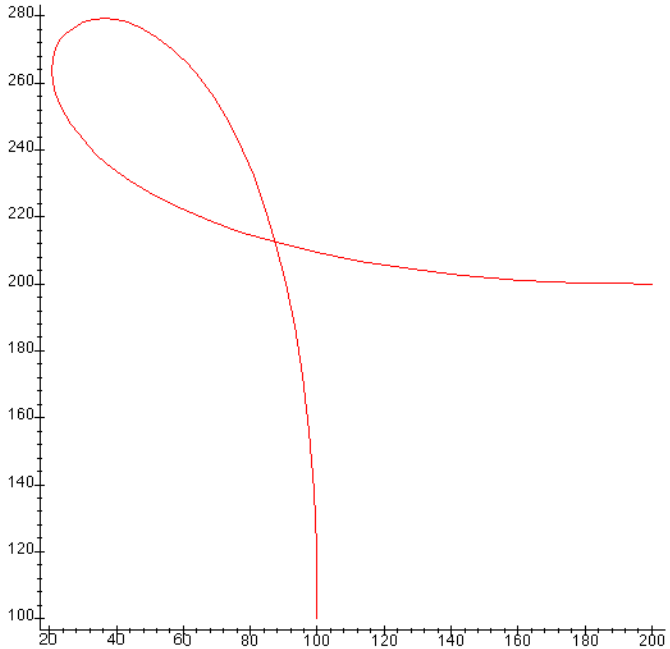
Examples

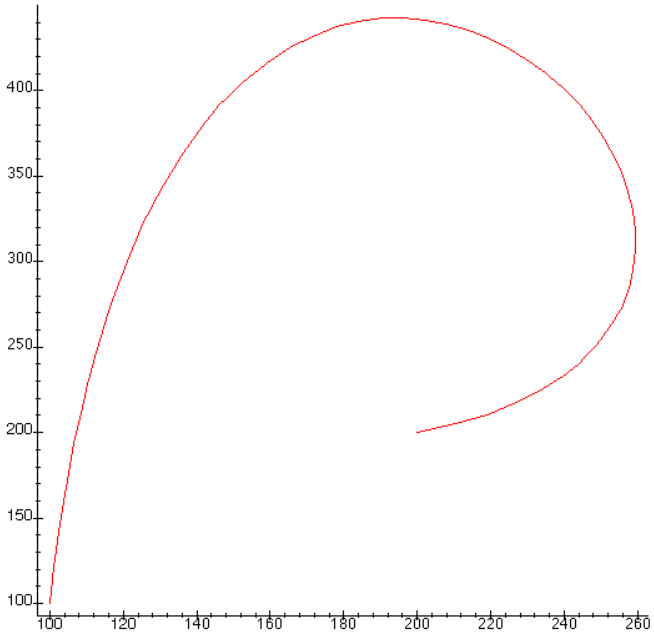




$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 10 & 10 & 0 \\ 10 & 10 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 0 & 200 & 0 \\ 200 & 0 & 0 \end{pmatrix}$$

More Examples





$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 0 & 1000 & 0 \\ 1000 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 100 & 2000 & 0 \\ -500 & -200 & 0 \end{pmatrix}$$

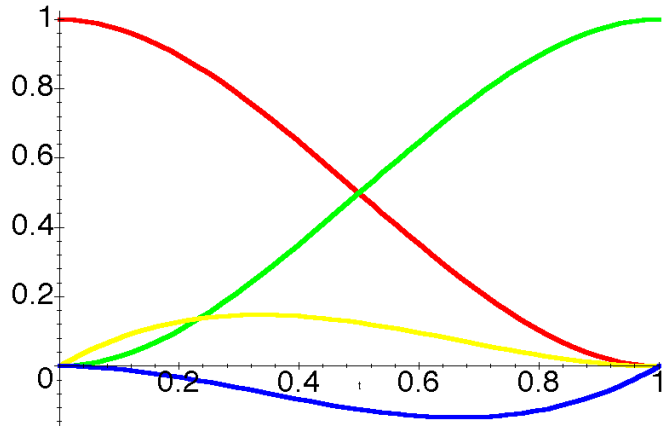
Hermite Blending Functions

- Conceptual Realignment
  - Curves are weighted averages of points/vectors.
  - Blending functions specify the weighting.

$$Q = \begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} (2\,t^3 - 3\,t^2 + 1) + \begin{bmatrix} x2 \\ y2 \\ z2 \end{bmatrix} (-2\,t^3 + 3\,t^2) + \begin{bmatrix} dx1 \\ dy1 \\ dz1 \end{bmatrix} (t^3 - 2\,t^2 + t) + \begin{bmatrix} dx2 \\ dy2 \\ dz2 \end{bmatrix} (t^3 - t^2)$$

$$Bh_1 = 2\,t^3 - 3\,t^2 + 1$$

$$Bh_2 = -2\,t^3 + 3\,t^2$$



$$Bh_3 = t^3 - 2\,t^2 + t$$

$$Bh_4 = t^3 - t^2$$

From Hermite to Bezier

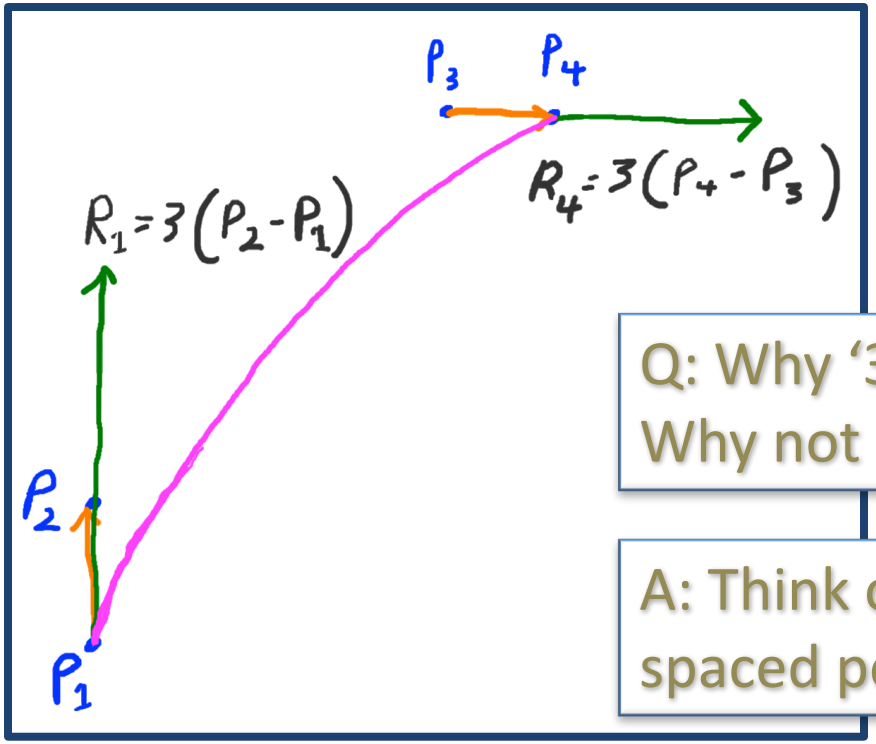
What's wrong with Hermite curves?

Nothing, unless you are using a point-and-click interface

Bezier curves are like Hermite curves, except that the user specifies four points (p_1, p_2, p_3, p_4).
The curve goes through p_1 & p_4.
Points p_2 & p_3 specify the tangents at the endpoints.

More Precisely....

$$R_1 = \frac{d}{dt}P_1 = 3(P_2 - P_1) \qquad R_4 = \frac{d}{dt}P_4 = 3(P_4 - P_3)$$



Tangents at start and end are now defined by intermediate points.

Q: Why '3'?
Why not $R_1=(P_2-P_1)$?

A: Think of 4 evenly spaced points in a line

Hermite → Bezier

The Hermite geometry matrix is related to the Bezier geometry matrix by:

$$G_H = \begin{pmatrix} P1 \\ P4 \\ R1 \\ R4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} P1 \\ P2 \\ P3 \\ P4 \end{pmatrix}$$

$$= M_{HB} G_B$$

Hermite $\rightarrow$ Bezier

For Hermite curves, $Q(t) = T\, M_H G_H$,
where $G_H = [P_1, P_4, R_1, R_4]^T$, $T = [t^3, t^2, t, 1]$

$$\text{and } M_H = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\boxed{\text{So, } Q(t) = T\, \underbrace{M_H M_{HB}} G_B}$$

The Bezier Basis Matrix

$$Q(t) = T(M_H M_{HB}) G_B$$

$$M_B = M_H M_{HB} = \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$Q(t) = T M_B G_B$$

See page 364 to connect with description in our textbook.

The Bezier Blending Functions

$$Q(t) = \begin{bmatrix} (-t^3 - 3t + 3t^2 + 1)\,x1 + (3t - 6t^2 + 3t^3)\,x2 + (3t^2 - 3t^3)\,x3 + t^3\,x4 \\ (-t^3 - 3t + 3t^2 + 1)\,y1 + (3t - 6t^2 + 3t^3)\,y2 + (3t^2 - 3t^3)\,y3 + t^3\,y4 \\ (-t^3 - 3t + 3t^2 + 1)\,z1 + (3t - 6t^2 + 3t^3)\,z2 + (3t^2 - 3t^3)\,z3 + t^3\,z4 \end{bmatrix}$$

$$\begin{aligned} Q(t) = {} & P_1(-t^3 + 3t^2 - 3t + 1) \\ & + P_2(3t^3 - 6t^2 + 3t) \\ & + P_3(-3t^3 + 3t^2) \\ & + P_4(t^3) \end{aligned}$$

Add them up.

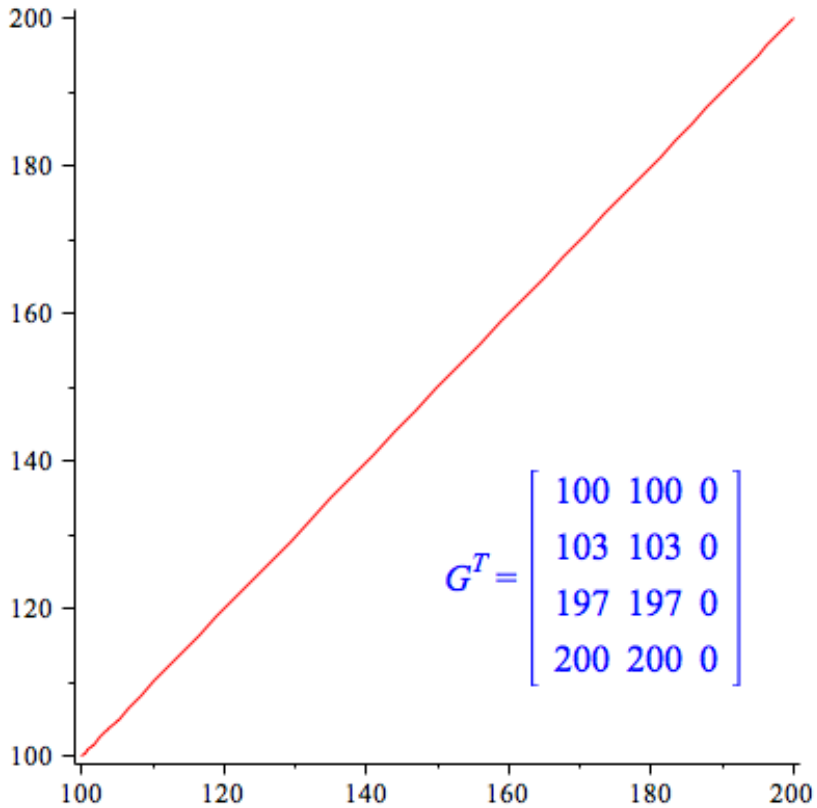
$$
\begin{array}{l}
\quad (-t^3 + 3t^2 - 3t + 1) \\
+ (3t^3 - 6t^2 + 3t) \\
+ (-3t^3 + 3t^2) \\
+ (t^3) \\
\hline
\quad 0t^3 + 0t^2 + 0t + 1
\end{array}
$$

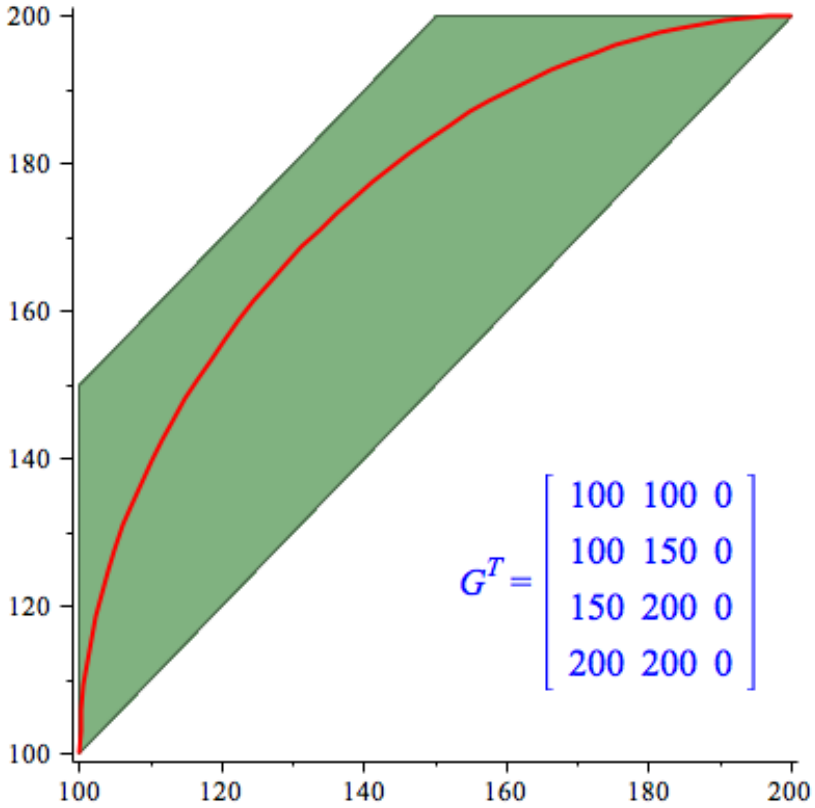
Stay within the Convex Hull

If you graph the four Bezier blending functions for t=0 to t=1, you find that they are always positive and always sum to one.

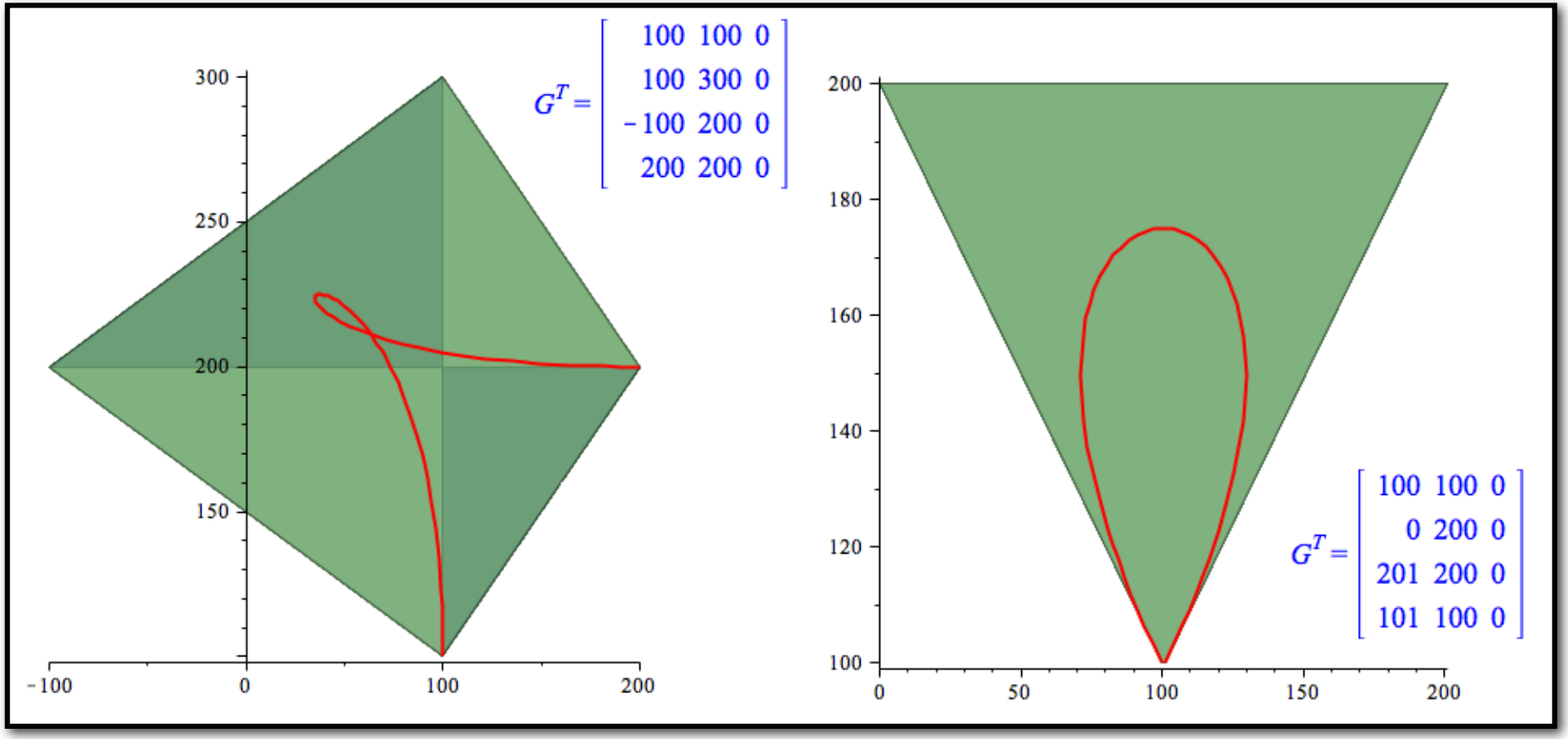
Therefore, *the Bezier curve stays within the convex hull defined by* P_1, P_2, P_3 & P_4.

Examples 1.


200
180
160
140
120
100
100
120
140
160
180
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 103 & 103 & 0 \\ 197 & 197 & 0 \\ 200 & 200 & 0 \end{bmatrix}$


200
180
160
140
120
100
100
120
140
160
180
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 100 & 150 & 0 \\ 150 & 200 & 0 \\ 200 & 200 & 0 \end{bmatrix}$

Examples 2.

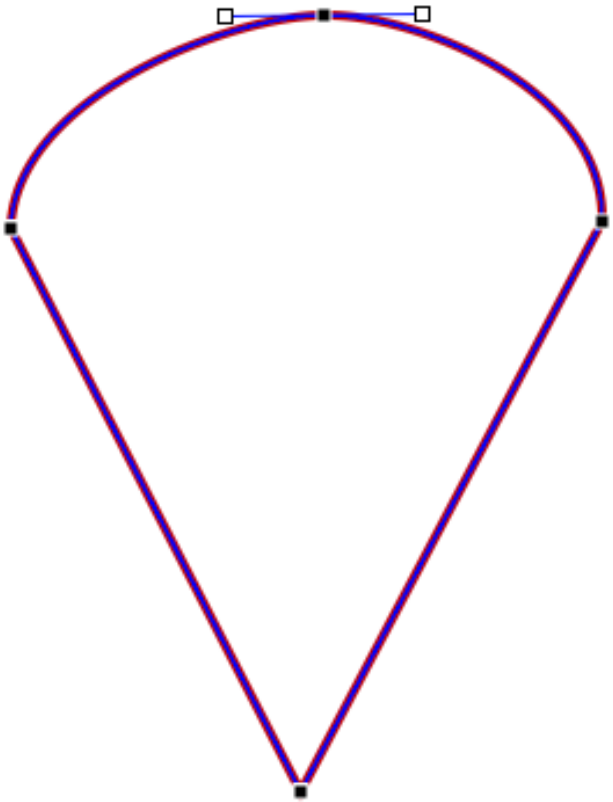

300
250
200
150
-100
0
100
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 100 & 300 & 0 \\ -100 & 200 & 0 \\ 200 & 200 & 0 \end{bmatrix}$
200
180
160
140
120
100
0
50
100
150
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 0 & 200 & 0 \\ 201 & 200 & 0 \\ 101 & 100 & 0 \end{bmatrix}$

Example 3D

```
P1 = (  0,   0,   0)
P2 = (100,   0,   0)
P3 = (  0, 100, 100)
P4 = (  0,   0, 100)
```

Bezier Curves are Common

- You have probably already used them.
- For example, in PowerPoint
  - Build a shape
  - Then select edit points
  - Notice the control ‘wings’
- Enhancements
  - Ways to introduce constraints
    - Smooth Point
    - Straight Point
    - Corner Point



Stepping Back

What should you be learning? Should you memorize M_H and M_B? No! That's what reference books are for.

You should know what G_H and G_B are. You should know how to derive M_H from the parametric form of the cubic equations. You should know how to derive M_B from M_H. If you understand these concepts, you can look up or rederive the matrices as necessary.

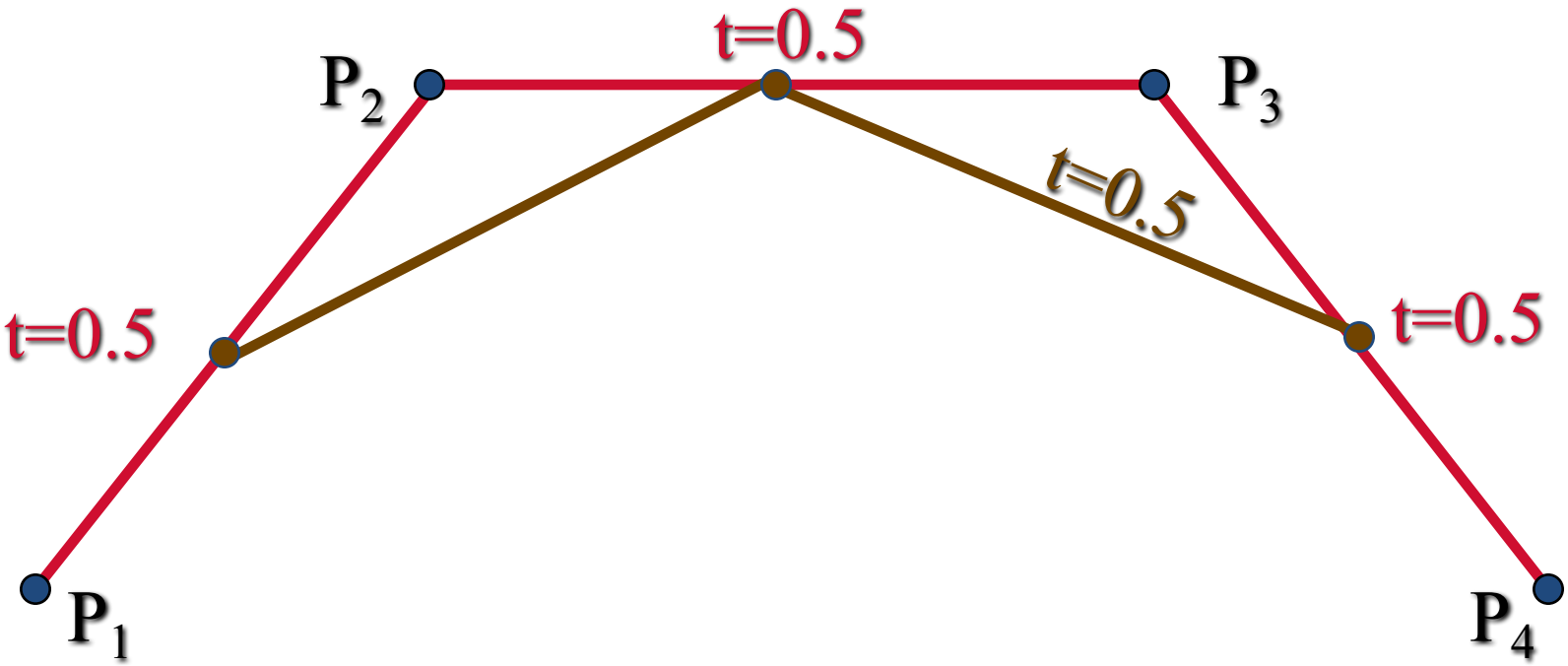
de Casteljau Curves

Now for something completely different.

There is another way to motivate curves.

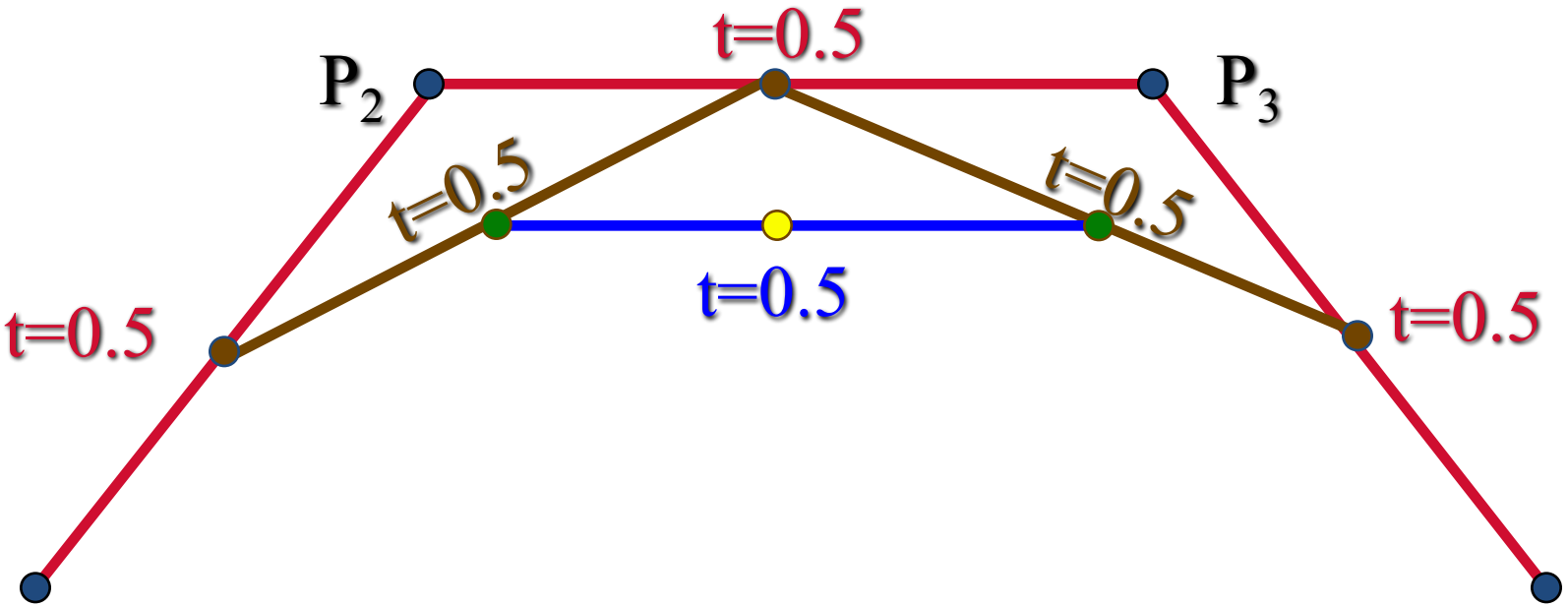
Lets say that I have four control points

To find the midpoint of the curve corresponding to those control points:



Connect the point between P_1 and P_2 where t = 0.5 with the point between P_3 & P_2 where t = 0.5;
Do the same with the P_2-P_3 & P_3-P_4 lines

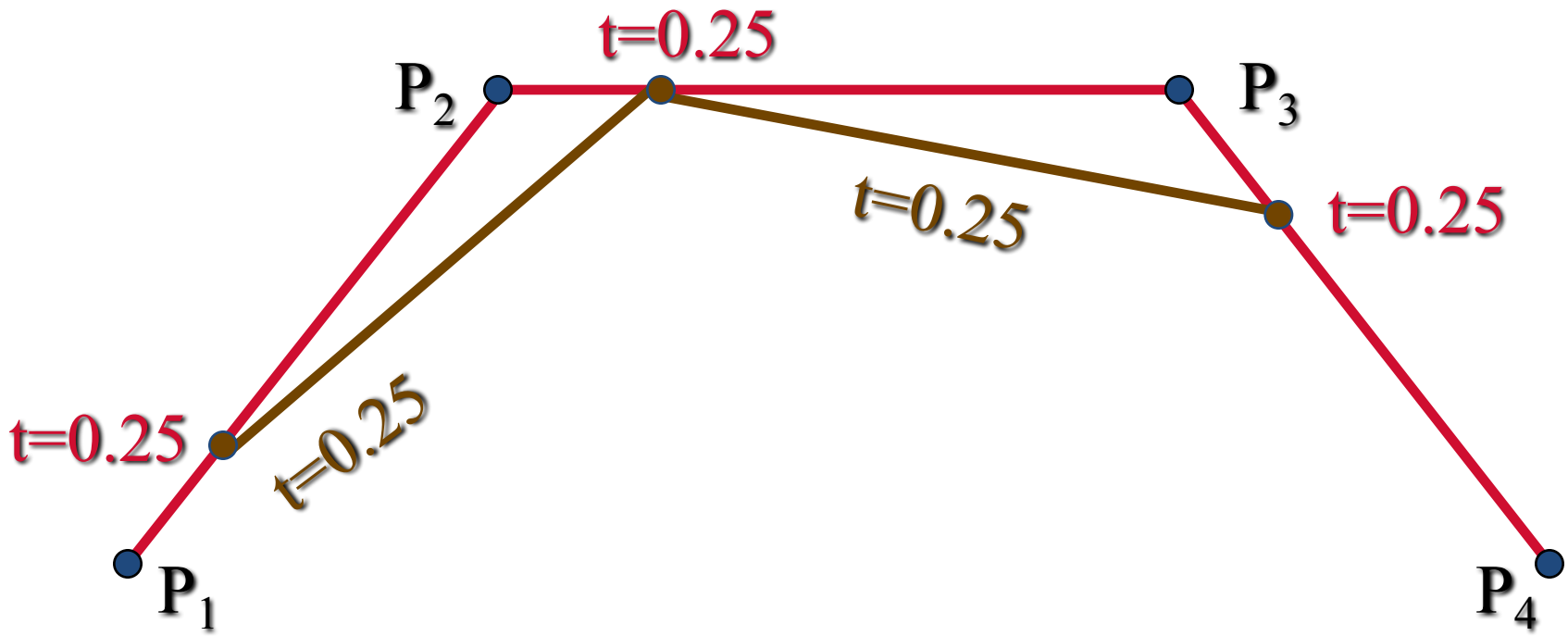
Now, connect these two lines at their t = 0.5 points



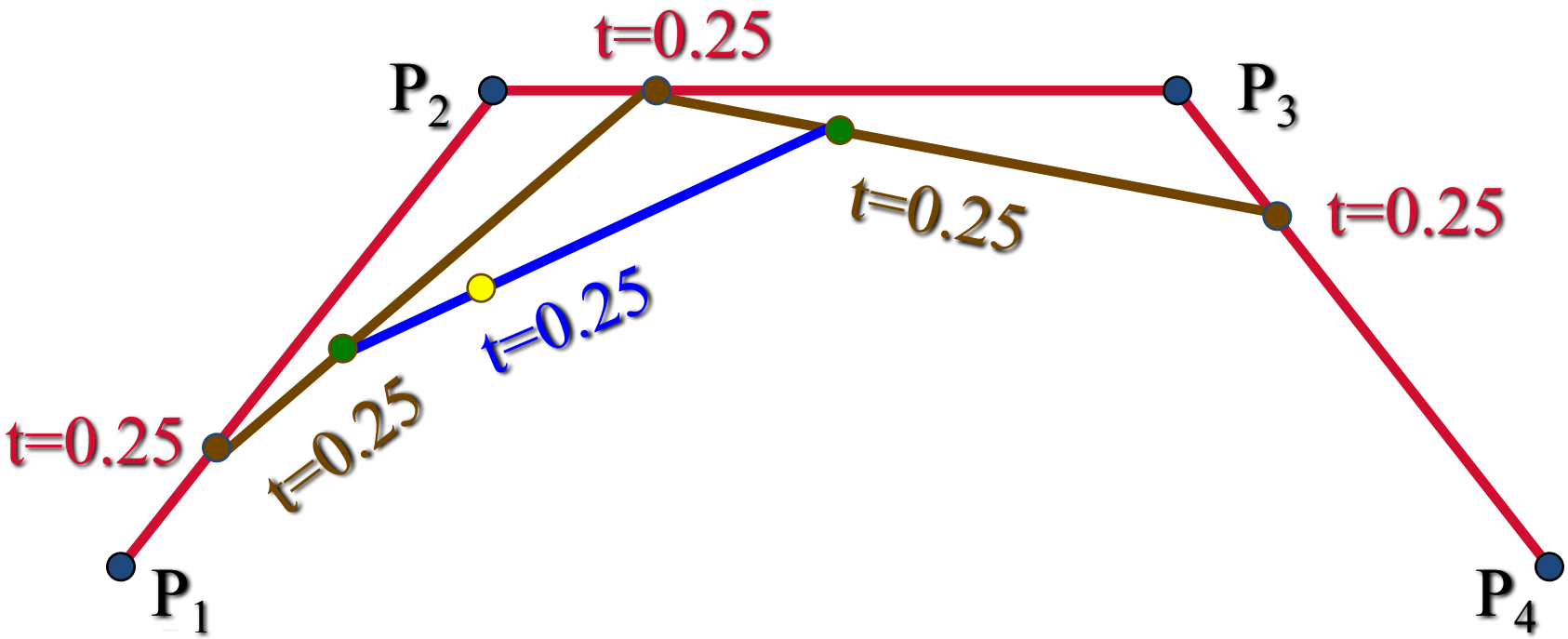
The t = 0.5 point on the resulting segment is the midpoint (t = 0.5 point) of some type of curve which is made up of weighted averages of the control points (*we'll soon see what kind of curve*)

We can extend this idea to any t value.

To compute the t = 0.25 point, connect the 0.25 points of the original lines...



Now, connect these lines at their t = 0.25 locations, and find the point where t = 0.25 of the resulting line



In this way, you can compute the 3rd-order curve for any value of t

Algebraic Definition

- The equations of the three original lines are:

$$A_1(t) = (1 - t)\, P_1 + t\, P_2$$

$$A_2(t) = (1 - t)\, P_2 + t\, P_3$$

$$A_3(t) = (1 - t)\, P_3 + t\, P_4$$

- The equations of the next two joining lines are:

$$B_1(t) = (1 - t)\, A_1 + t\, A_2$$

$$B_2(t) = (1 - t)\, A_2 + t\, A_3$$

- Finally, the line between the two joining lines is:

$$C_1(t) = (1 - t)\, B_1 + t\, B_2$$

Begin Substitutions

- Substitute equations for A_1 and A_2 into B_1

$$B_1(t) = (1-t)\,((1-t)\,P_1 + t\,P_2) + t\,((1-t)\,P_2 + t\,P_3)$$

$$= P_1 - 2\,t\,P_1 + t^2\,P_1 + 2\,t\,P_2 - 2\,t^2\,P_2 + t^2\,P_3$$

$$= (t^2 + 1 - 2\,t)\,P_1 + (-2\,t^2 + 2\,t)\,P_2 + t^2\,P_3$$

Factoring the result

$$B_1(t) = (t-1)^2\,P_1 - 2\,t\,(t-1)\,P_2 + t^2\,P_3$$

- Likewise, substitute A_2 and A_3 into B_2

$$B_2(t) = (t-1)^2\,P_2 - 2\,t\,(t-1)\,P_3 + t^2\,P_4$$

Resulting Third Order Curve

Substitute equations for B_1 and B_2 into C_1

$$C_1(t) = (1-t)\left((t-1)^2 P_1 - 2\,t\,(t-1)\,P_2 + t^2 P_3\right)$$
$$+\, t\left((t-1)^2 P_2 - 2\,t\,(t-1)\,P_3 + t^2 P_4\right)$$

$$= \left(-3\,t - t^3 + 3\,t^2 + 1\right) P_1$$
$$+ \left(3\,t^3 - 6\,t^2 + 3\,t\right) P_2$$
$$+ \left(-3\,t^3 + 3\,t^2\right) P_3 + t^3 P_4$$

And After Factoring

$$C_1(t) = (1-t)^3 P_1 + 3\,t\,(t-1)^2 P_2 - 3\,t^2\,(t-1)\,P_3 + t^3 P_4$$

Bezier = de Casteljau

But those last four functions are exactly the Bezier blending functions!

The recursive line intersection algorithm can therefore be used to gain intuition about the behavior of Bezier functions

Not something completely different afterall.