Lecture 14: Perspective Projection

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3D Viewing as Virtual Camera

To take a picture with a camera, or to render an image with computer graphics, we need to:

1. Position the camera/viewpoint in 3D space
2. Orient the camera/viewpoint in 3D space
3. Transform objects into Camera Coordinates
4. Crop scene(objects to frustum
5. Project remaining objects to the image plane
Perspective ...
If not for the fog, you could see forever …

and nothing ever would look smaller.
Orthographic / Perspective
Think About Rays
Is Perspective Always Better?

No! Technical programs, including for example Maple, often favor orthographic projection.
Math: Orthographic Projection

• Simply drop a dimension.

\[
\begin{bmatrix}
  u \\
  v \\
  0 \\
  1 \\
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1 \\
\end{bmatrix}
\]

• Think of a bug hitting a windshield.

• No more z axis!
  – no more bug

Photo by Brian, Jeff Booth site
www.jeffbooth.net (creative common License)
Perspective Projection

- Light rays pass through the focal point.
  - a.k.a. Eye, principal reference point, or PRP.
- The image plane is an infinite plane in front of (or behind) the focal point.
- Images are formed by rays of light passing through the image plane.
- Common convention:
  - Image points are \((u,v)\)
  - World points are \((x,y,z)\)
Why “Pinhole” Camera?

• Because you can build a camera that exactly fits this description:
  – Create a fully-enclosed black box
    • So that no light enters
  – Put a piece of film inside it, facing front
  – Punch a pin-hole in the front face of the box

• What doesn’t this camera have?
• What is this camera’s depth-of-field?
• Why don’t we build cameras this way?
History

• The Camera Obscura - see Wikipedia


• Pre-dates photographic cameras.
  – Theory: Mo-Ti (China, 470-390 BC)
  – Practice: Abu Ali Al-Hasan Ibn al-Haitham (~1000 AD)
  – Western Painting: Johannes Vermeer (~1660 AD)
Pinhole Projection
Flip the Bear in the Box
Human Eye - 4 year old view

Drawing by Bryce Beveridge in 2006
Room Obscura

You Can Do It! Room Obscura
230,986 views
Perspective Projection

• Where we place the origin matters
• How we handle z values matters
• Form #1:
  – Origin at focal point, z values constant
• Form #2:
  – Origin at image center, z values are zero
• Form #3: *(next lecture)*
  – Origin at focal point, z proportional to depth
The key to perspective projection is that all light rays meet at the PRP (E, focal point).

Notice that we are looking down the Z axis, with the origin at the focal point and the image plane at $z = d$. 

\[
P(x,y,z)
\]

\[
P_y
\]

\[
P_z
\]

\[
d
\]
By similar triangles:

\[
\frac{P_u}{d} = \frac{P_x}{P_z}
\]

\[
P_u = P_x \frac{d}{P_z}
\]

\[
\frac{P_v}{d} = \frac{P_y}{P_z}
\]

\[
P_v = P_y \frac{d}{P_z}
\]
Perspective Projection Matrix

Problem: division of one variable by another is a non-linear operation.

Solution: homogeneous coordinates!

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1/d & 0 \\
\end{vmatrix}
\]
Perspective Matrix (II)

\[
\begin{bmatrix}
u \\
v \\
d \\
d
\end{bmatrix} = \begin{bmatrix}d & x \\
z & y \\
z/d & 1 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1/d & 0
\end{bmatrix} \begin{bmatrix}x \\
y \\
z \\
1
\end{bmatrix}
\]

Point in (u,v) coordinates

Point in Non-normalized Homogeneous coordinates

Normalized

Projection Matrix times a Point
What happens to Z?

• What happens to the Z dimension?

\[
\begin{bmatrix}
  u \\
  v \\
  d \\
  1 \\
\end{bmatrix}
= \begin{bmatrix}
  \frac{d}{z} \\
  \frac{x}{z} \\
  \frac{y}{z} \\
  \frac{z}{d} \\
\end{bmatrix}
= \begin{bmatrix}
  x \\
  y \\
  z \\
  z/d \\
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 1/d & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1 \\
\end{bmatrix}
\]

• The Z dimension projects to d. Why?
• Because \((u, v, d)\) is a 3D point on the image plane located at \(z = d\)!
Perspective Projection Form #2

\[ \frac{v}{d} = \frac{P_y}{d + P_z} \quad v = P_y \left( \frac{d}{d + P_z} \right) \]
Leading to the following

\[
\begin{bmatrix}
  x \left( \frac{d}{d+z} \right) \\
  y \left( \frac{d}{d+z} \right) \\
  0 \\
  1
\end{bmatrix} =
\begin{bmatrix}
  x \\
  y \\
  0 \\
  \frac{z+d}{d}
\end{bmatrix} =
\begin{bmatrix}
  x \\
  y \\
  0 \\
  \frac{z}{d} + 1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \frac{1}{d} & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]

• Now look at what happens to depth.

• Contrast this with previous version.
Let distance $d$ go to infinity.

**Formulation #1**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{d} & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
z \\
\frac{z}{d}
\end{bmatrix}
\cdot \begin{bmatrix}
\frac{1}{z/d} \\
0 \\
0 \\
1
\end{bmatrix}
= \frac{d}{z}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

**Formulation #2**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1 + \frac{z}{d}
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
z \\
1 + \frac{z}{d}
\end{bmatrix}
\cdot \begin{bmatrix}
\frac{1}{1 + \frac{z}{d}} \\
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
\frac{x}{1 + \frac{z}{d}} \\
\frac{y}{1 + \frac{z}{d}} \\
0 \\
1
\end{bmatrix}
\]

Recall formulation #2 when considering how projection changes with increased focal length.