Lecture 14: Perspective Projection

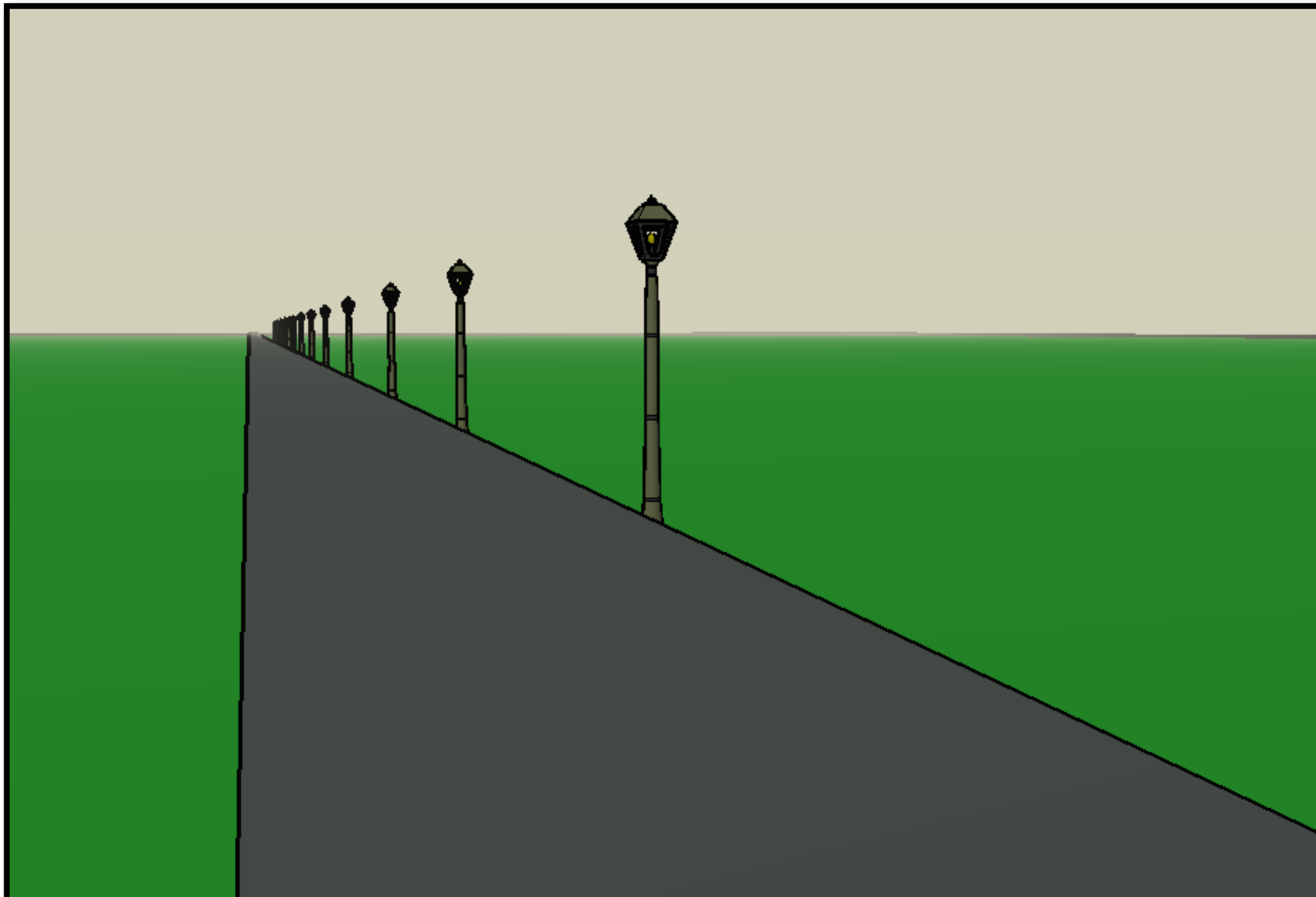
October 17, 2019

3D Viewing as Virtual Camera

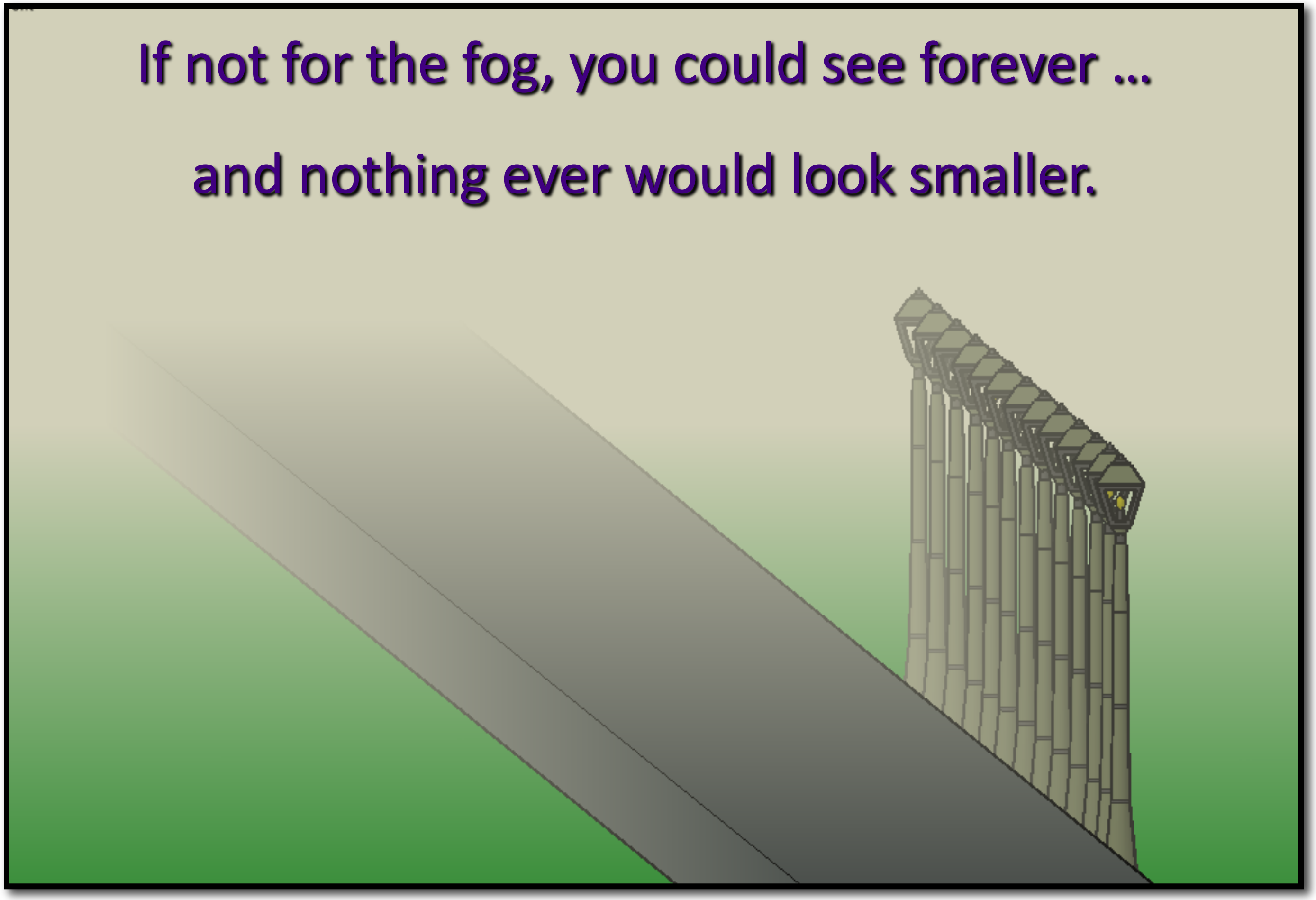
To take a picture with a camera, or to render an image with computer graphics, we need to:

1. Position the camera/viewpoint in 3D space
2. Orient the camera/viewpoint in 3D space
3. Transform objects into Camera Coordinates
4. Crop scene/objects to frustum
5. Project remaining objects to the image plane

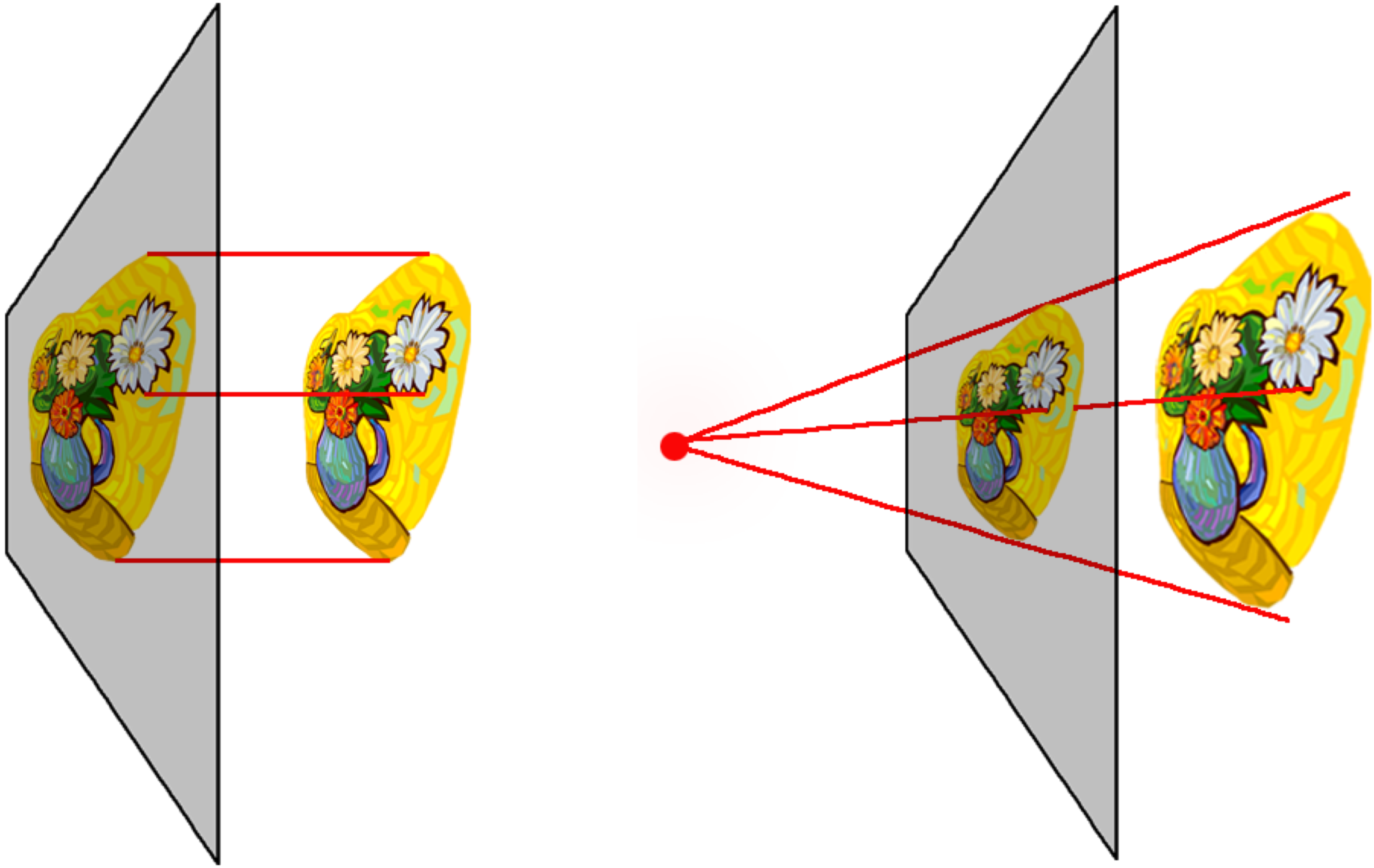
Perspective …



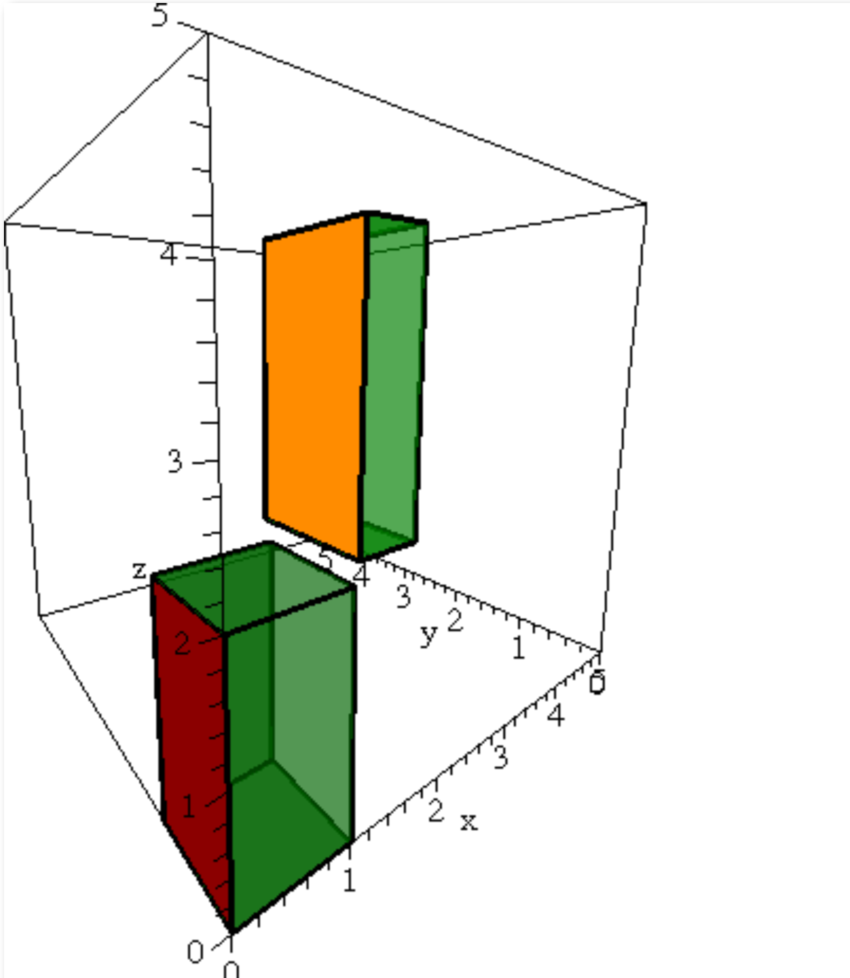
Orthographic Projection

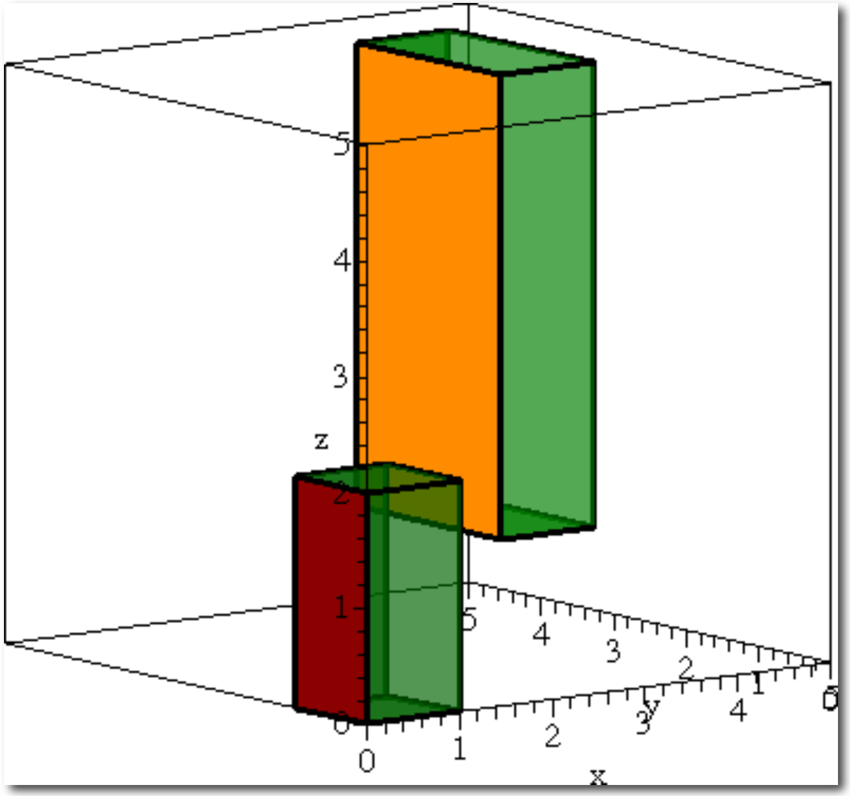


Orthographic / Perspective
Think About Rays



Is Perspective Always Better?





No! Technical programs, including for example Maple, often favor orthographic projection.

Math: Orthographic Projection

- Simply drop a dimension.

$$\begin{bmatrix} u \\ v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Think of a bug hitting a windshield.
- No more z axis!
  - *no more bug*

Photo by Brian, Jeff Booth site
www.jeffbooth.net (creative common License)

Perspective Projection

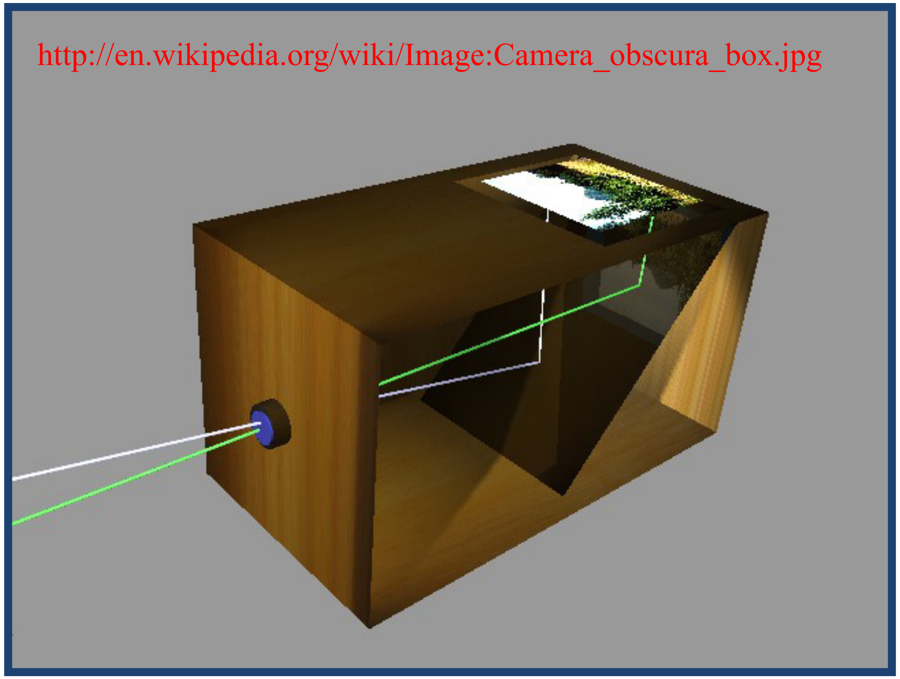
- Light rays pass through the focal point.
  - a.k.a. Eye, principal reference point, or PRP.
- The image plane is an infinite plane in front of (or behind) the focal point.
- Images are formed by rays of light passing through the image plane
- Common convention:
  - Image points are (u,v)
  - World points are (x,y,z)

Why "Pinhole" Camera?

- Because you can build a camera that exactly fits this description:
  - Create a fully-enclosed black box
    - So that no light enters
  - Put a piece of film inside it, facing front
  - Punch a pin-hole in the front face of the box
- What doesn't this camera have?
- What is this camera's depth-of-field?
- Why don't we build cameras this way?

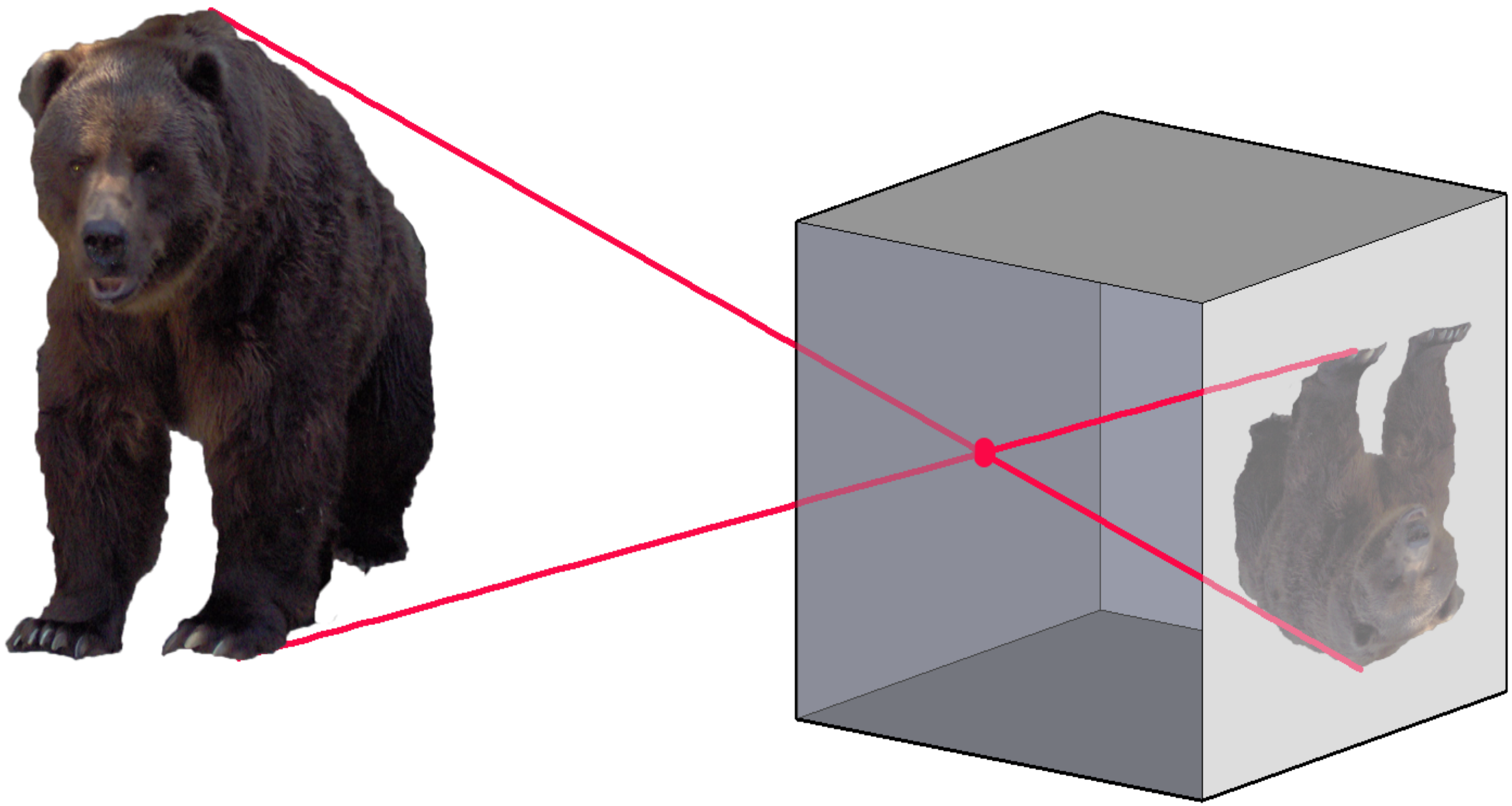
History

- The Camera Obscura - see Wikipedia

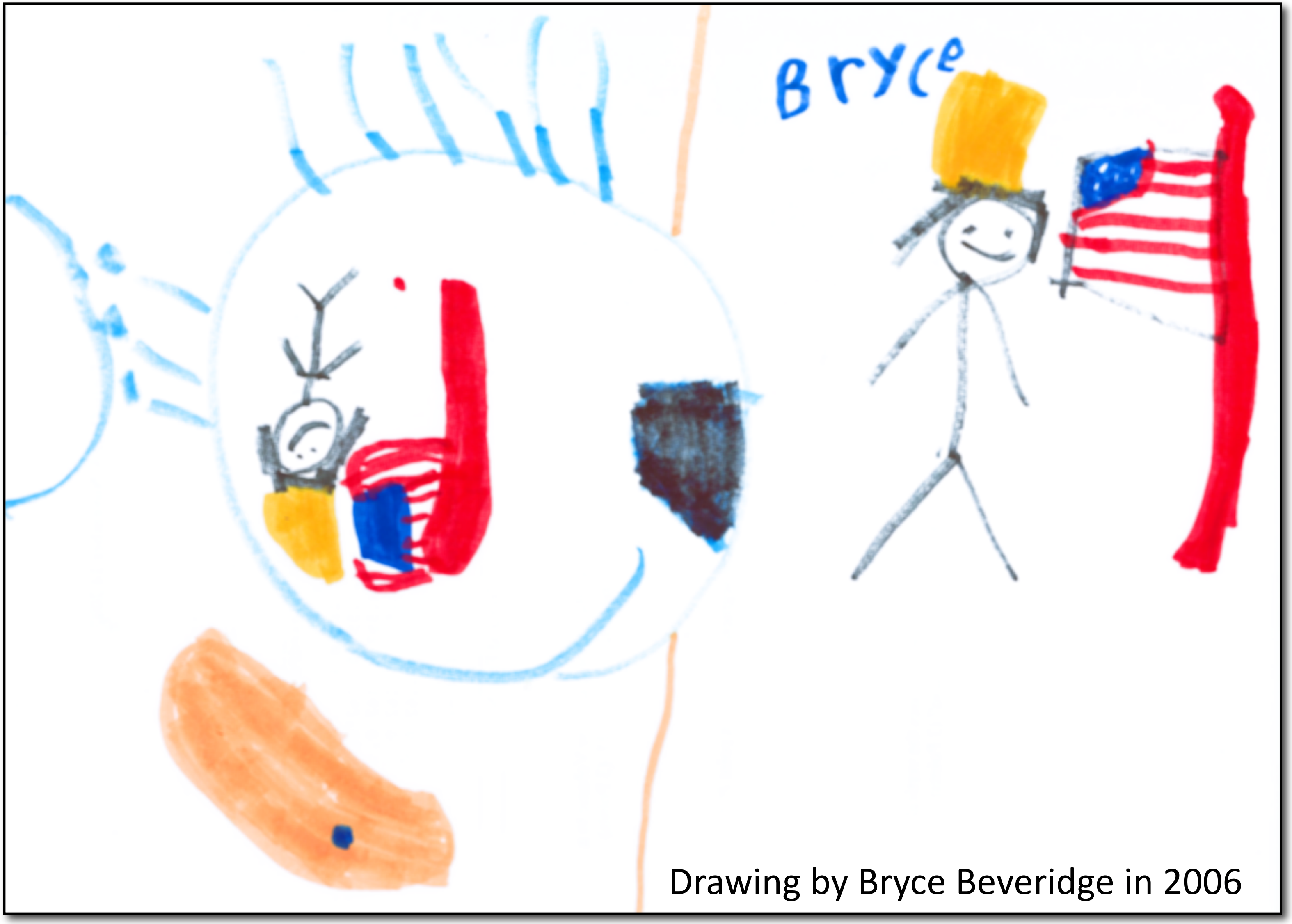


- Pre-dates photographic cameras.
  - Theory: Mo-Ti (China, 470-390 BC)
  - Practice: Abu Ali Al-Hasan Ibn al-Haitham (~1000 AD)
  - Western Painting: Johannes Vermeer (~1660 AD)

Pinhole Projection
Flip the Bear in the Box



Human Eye - 4 year old view



Drawing by Bryce Beveridge in 2006

Room Obscura

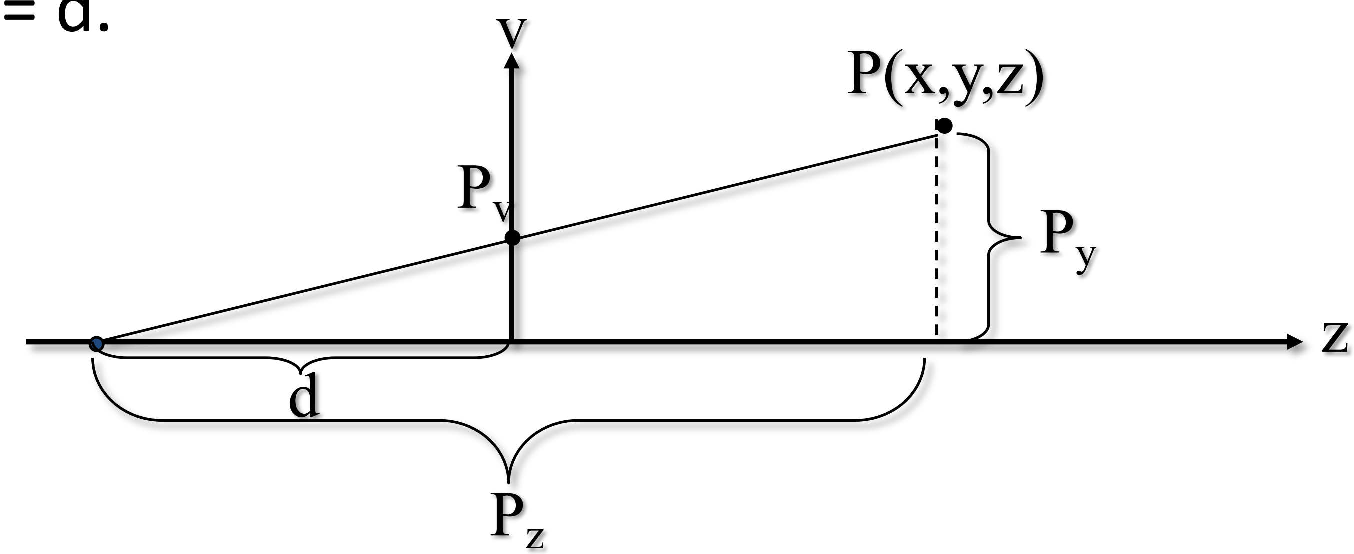
Perspective Projection

- Where we place the origin matters
- How we handle z values matters
- Form #1:
  - Origin at focal point, z values constant
- Form #2:
  - Origin at image center, z values are zero
- Form #3: (*next lecture*)
  - Origin at focal point, z proportional to depth

Perspective Projection Form #1

The key to perspective projection is that all light rays meet at the PRP (E, focal point).

Notice that we are looking down the Z axis, with the origin at the focal point and the image plane at z = d.



By similar triangles:

horizontal

$$\frac{P_u}{d} = \frac{P_x}{P_z}$$

$$P_u = \frac{P_x d}{P_z}$$

$$P_u = P_x \frac{d}{P_z}$$

vertical

$$\frac{P_v}{d} = \frac{P_y}{P_z}$$

$$P_v = \frac{P_y d}{P_z}$$

$$P_v = P_y \frac{d}{P_z}$$

Perspective Projection Matrix

Problem: division of one variable by another is a non-linear operation.

Solution: homogeneous coordinates!

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{vmatrix}$$

Perspective Matrix (II)

$$\begin{vmatrix} u \\ v \\ d \\ 1 \end{vmatrix} = \begin{vmatrix} x\dfrac{d}{z} \\ y\dfrac{d}{z} \\ d \\ 1 \end{vmatrix} = \begin{vmatrix} x \\ y \\ z \\ {}^{z}/_{d} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & {}^{1}/_{d} & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

Point in (u,v) coordinates

Normalized

Point in Non-normalized Homogeneous coordinates

Projection Matrix times a Point

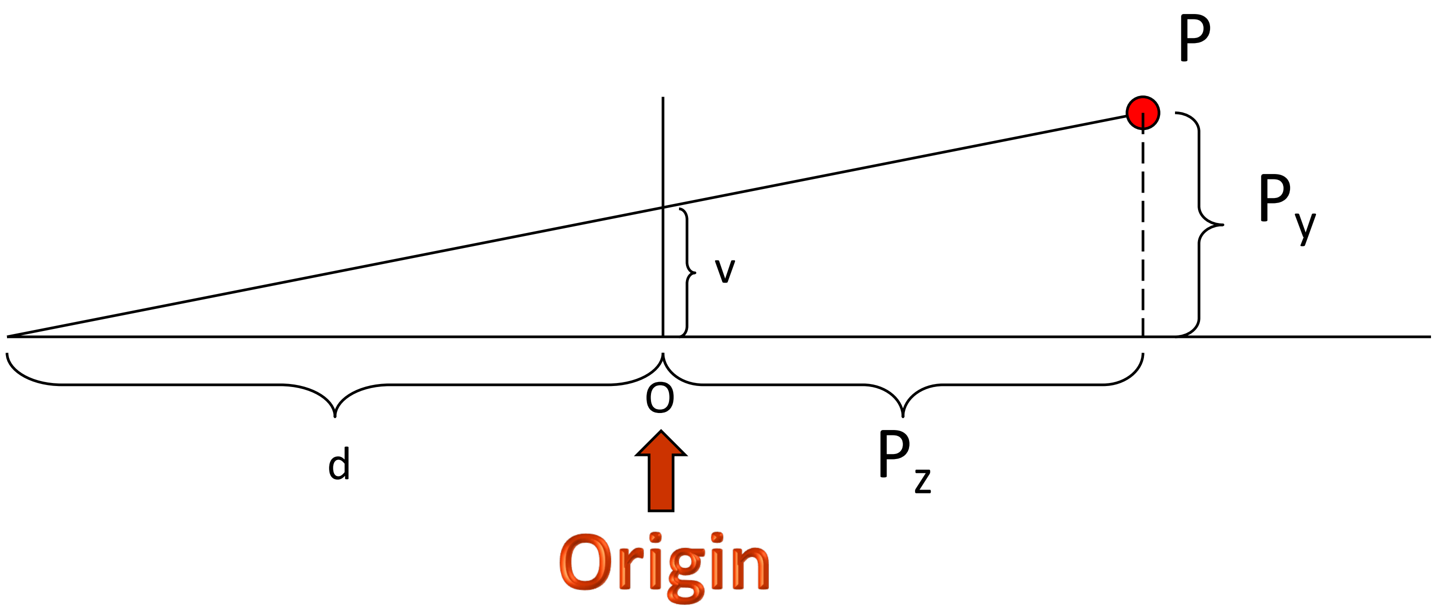
What happens to Z?

- What happens to the Z dimension?

$$\begin{vmatrix} u \\ v \\ d \\ 1 \end{vmatrix} = \begin{vmatrix} x\dfrac{d}{z} \\ y\dfrac{d}{z} \\ d \\ 1 \end{vmatrix} = \begin{vmatrix} x \\ y \\ z \\ z/d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

- The Z dimension projects to d. Why?
- Because (u, v, d) is a 3D point on the image plane located at z = d!

Perspective Projection Form #2



$$\frac{v}{d} = \frac{P_y}{d + P_z} \qquad v = P_y\left(\frac{d}{d + P_z}\right)$$

Leading to the following

$$\begin{bmatrix} x\left(\frac{d}{d+z}\right) \\ y\left(\frac{d}{d+z}\right) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ \frac{z+d}{d} \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ \frac{z}{d}+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Now look at what happens to depth.
- Contrast this with previous version.

Let distance d go to infinity.

Formulation #1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix} \qquad \left(\frac{1}{\frac{z}{d}}\right) \cdot \begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix} = \begin{bmatrix} \frac{d\,x}{z} \\ \frac{d\,y}{z} \\ d \\ 1 \end{bmatrix}$$

Formulation #2

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 + \frac{z}{d} \end{bmatrix} \qquad \left(\frac{1}{1 + \frac{z}{d}}\right) \cdot \begin{bmatrix} x \\ y \\ 0 \\ 1 + \frac{z}{d} \end{bmatrix} = \begin{bmatrix} \frac{x}{1 + \frac{z}{d}} \\ \frac{y}{1 + \frac{z}{d}} \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{vmatrix} X \\ Y \\ 0 \\ 1 \end{vmatrix}$$

Recall formulation #2 when considering how projection changes with increased focal length.