# Lecture 23:
# Hermite and Bezier Curves

December 3, 2019

# Representing Curved Objects

- So far we've seen
  - Spheres
  - Polygonal objects (triangles)
- Now, polynomial curves
  - Hermite curves
  - Bezier curves
  - B-Splines
  - NURBS
- Bivariate polynomial surface patches

# Beyond linear approximation

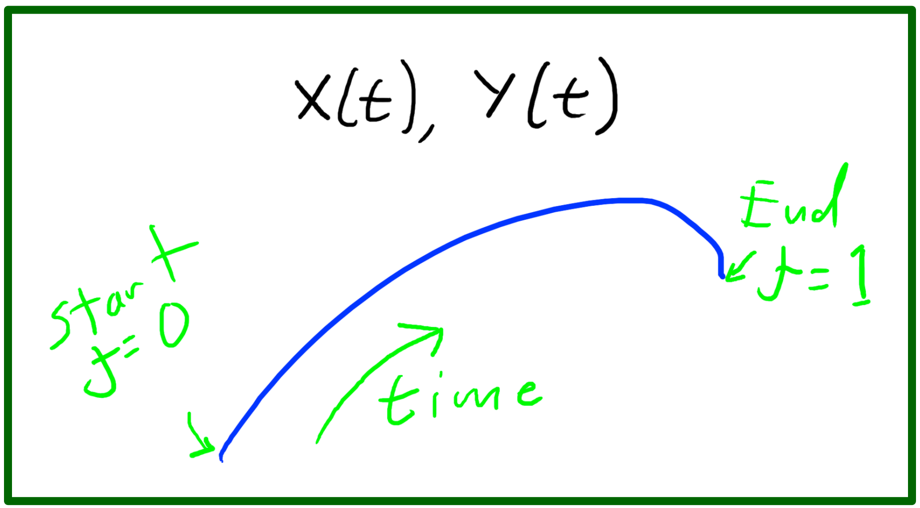
Instead of approximating everything by zillions of lines and planes, it is possible to approximate shapes using higher-order curves. Advantages:

- More compact
- Reduces “artifacts”

*Use of sphere in ray tracer is an example of an implicit curve.*

# The Pen Metaphore

- Think of putting a pen to paper
- Pen position described by time t




Seeing the action of drawing is the key, so this static drawing only partly captures the point of this slide.

# Design Criteria

- Local control of shape
- Smoothness and continuity
- Ability to evaluate derivatives
- Stability
- Ease of rendering

# Review Forms - Explicit

- Explicit representation: y = f(x)

$$y = ax^3 + bx^2 + cx + d$$

- Drawbacks:
  - Multiple values of y for a single x impossible.
  - Not rotationally invariant.

# Review Forms - Implicit

- Implicit representation: f(x, y, z) = 0

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

- Advantages:
  - – On curve & relative distance to curve tests.
- Drawbacks:
  - – Enumerating points on the curve is hard.
  - – Extra constraints needed – half a circle?
  - – Difficult to express and test tangents.

# Parametric Representations

We will represent 3D curves using a parametric representation, introducing a new variable t:

$$Q(t) = \left| \; x(t) \quad y(t) \quad z(t) \; \right|$$

Note that x, y and z are dependent on t alone, making it clear that there is only one free variable.

Think of t as time associated with movement along the curve.

# Third Order Curves

- Third-order functions are the standard:

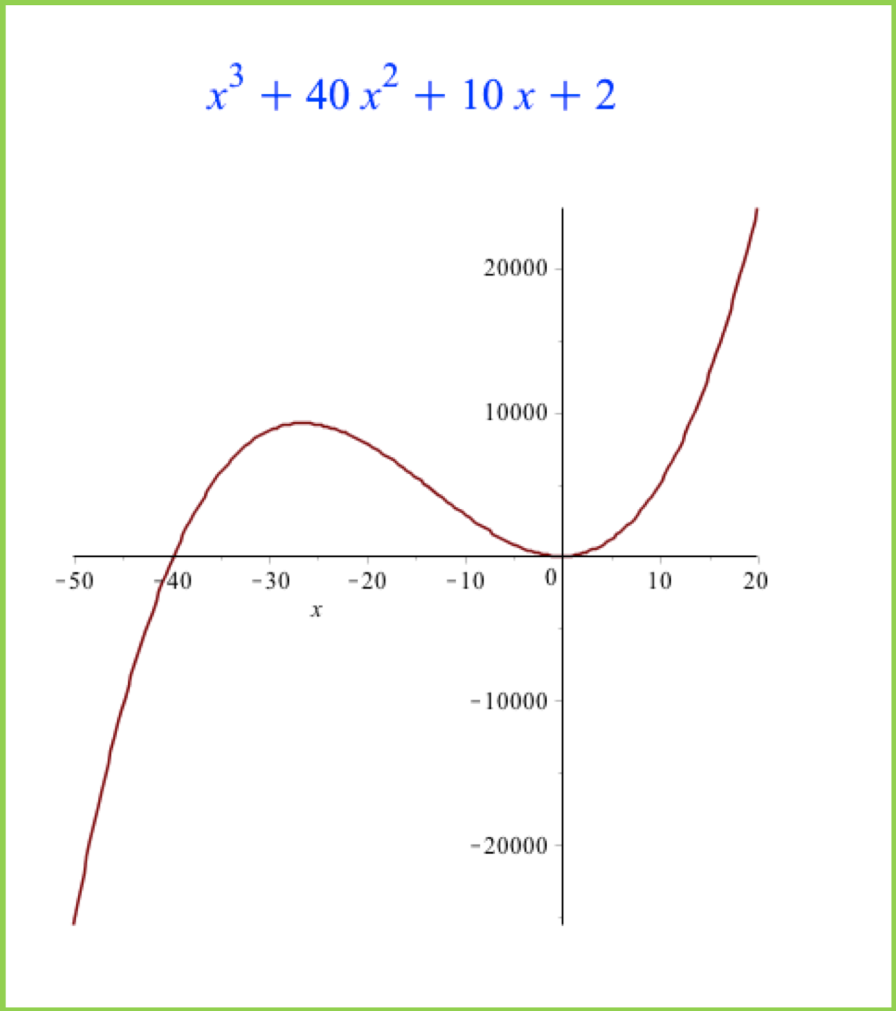
$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

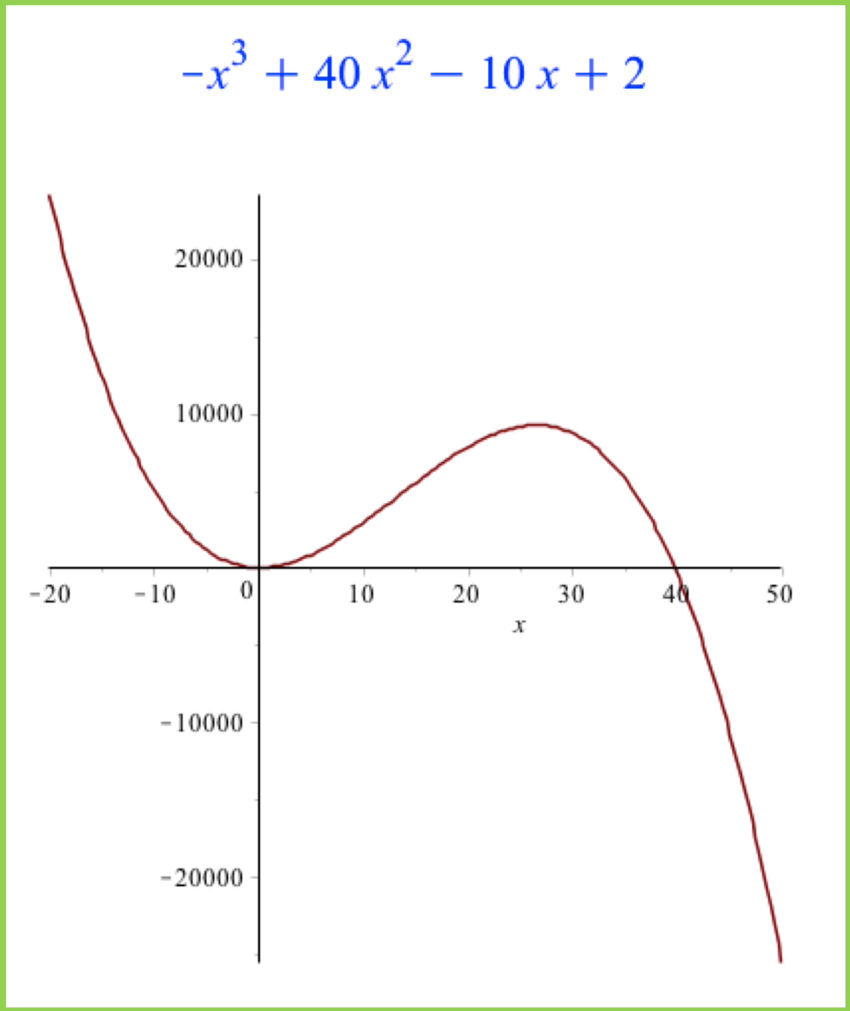
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

- Why 3?
  - Lower-order curves cannot be smoothly joined.
  - Higher-order curves introduce “wiggles”.
- Without loss of generality: 0 <= t <= 1.

# Cubic Examples



# Notation

$$T = \left[t^3, t^2, t, 1\right]$$

$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

$$Q(t) = T \cdot C$$

Alternatively: $Q(t)^T = C^T \cdot T^T$

# Tangents to Cubic Curves

The derivative of Q(t) is its tangent:

$$\frac{d}{dt}\ Q(t) = [\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t)]$$

$$\frac{d}{dt}Q(t)_x = 3a_x t^2 + 2b_x t + c_x$$

$$\frac{d}{dt}Q(t) = \quad [3t^2, 2t, 1, 0]\ C$$

The same matrix as on previous slide

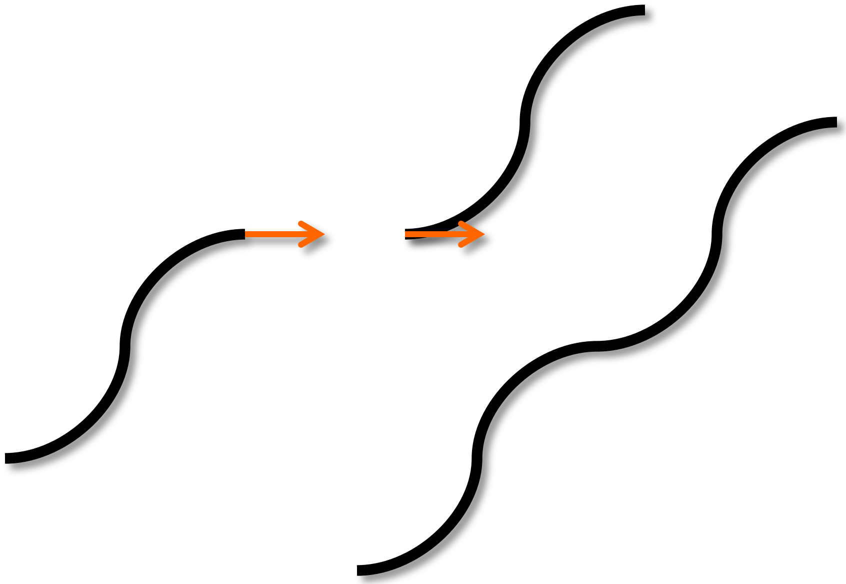
Again the time metaphor is useful, the tangent indicates instantaneous direction and speed.

# Hermite Curves

We want curves that fit together smoothly.

To accomplish this, we would like to specify a curve by providing:

- The endpoints
- The 1st derivatives at the endpoints



The result is called a *Hermite Curve*.

# Hermite Curves (cont.)

Since Q(t) = TC, we factor C into two matrices:

G (a 3x4 geometry matrix)

M (a 4x4 basis matrix)

such that C = G • M.

This step is a big deal. It makes thinking about curve geometry tractable.

*Note: G will hold our geometric constraints (endpoints and derivatives), while M will be constant across all Hermite curves.*

Let us concentrate on the x component:

$$P(t)_x = a_x t^3 + b_x t^2 + c_x t + d_x$$

Remember that its derivative is:

$$\frac{d}{dt} P(t)_x = 3a_x t^2 + 2b_x t + c_x$$

Therefore

$$\begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\ P(0)_x \\ d/dt\ P(1)_x \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix}$$

Therefore:

$$\begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix}^{-1} \begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\ P(0)_x \\ d/dt\ P(1)_x \end{vmatrix}$$

And taking the inverse:

$$\begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} P(0)_x \\ P(1)_x \\ d/dt\ P(0)_x \\ d/dt\ P(1)_x \end{vmatrix}$$

# The Hermite Matrix

OK, that was the x dimension.
How about the others?

They are, of course, the same:

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{vmatrix} = \underbrace{\begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}}_{M} \begin{vmatrix} P(0) \\ P(1) \\ \frac{dP(0)}{dt} \\ \frac{dP(1)}{dt} \end{vmatrix}$$

# Punchline

$$Q(t) = \begin{vmatrix} x(t) & y(t) & z(t) \end{vmatrix} = T \cdot C = T \cdot M_H \cdot \begin{bmatrix} P(0) \\ P(1) \\ {}^{d}\!/_{dt} P(0) \\ {}^{d}\!/_{dt} P(1) \end{bmatrix}$$

*Since $M_H$ and T are known, you can write down a cubic polynomial curve by inspection ending at points P(0) and P(1) with tangents d/dt P(0) and d/dt P(1).*

# Expand the Math – Look Inside

Recall Parametric Equation: $Q(t) = T\, M_H\, G$

Where

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x1 & y1 & z1 \\ x2 & y2 & z2 \\ dx1 & dy1 & dz1 \\ dx2 & dy2 & dz2 \end{bmatrix}$$

Fully Expanded (note transpose)

$$Q(t)^T = \begin{bmatrix} (2t^3 - 3t^2 + 1)\, x1 + (3t^2 - 2t^3)\, x2 + (-2t^2 + t^3 + t)\, dx1 + (-t^2 + t^3)\, dx2 \\ (2t^3 - 3t^2 + 1)\, y1 + (3t^2 - 2t^3)\, y2 + (-2t^2 + t^3 + t)\, dy1 + (-t^2 + t^3)\, dy2 \\ (2t^3 - 3t^2 + 1)\, z1 + (3t^2 - 2t^3)\, z2 + (-2t^2 + t^3 + t)\, dz1 + (-t^2 + t^3)\, dz2 \end{bmatrix}$$

# If you Prefer

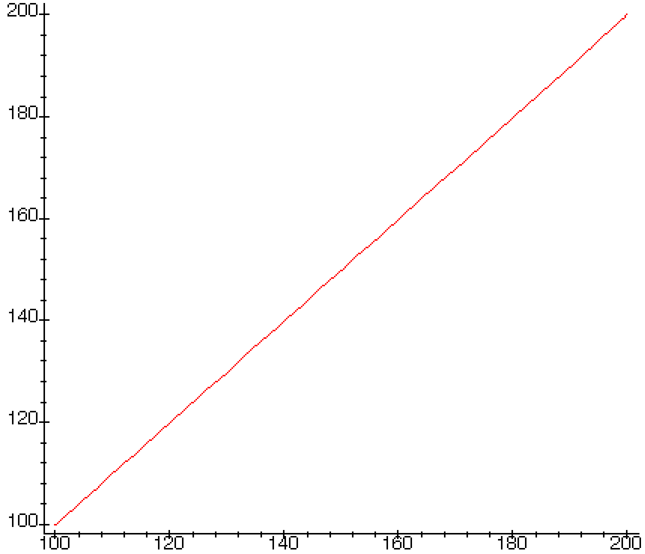
- There are two equivalent setups
- The difference is solely transposition

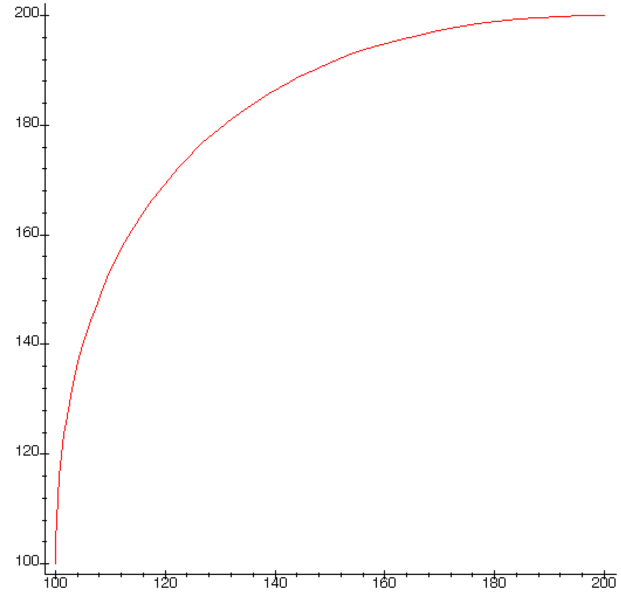
$$Q(t) = G\,M\,T$$

$$Q(t) = \begin{bmatrix} x1 & x2 & dx1 & dx2 \\ y1 & y2 & dy1 & dy2 \\ z1 & z2 & dz1 & dz2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} \left(2\,t^3 - 3\,t^2 + 1\right) x1 + \left(3\,t^2 - 2\,t^3\right) x2 + \left(-2\,t^2 + t^3 + t\right) dx1 + \left(-t^2 + t^3\right) dx2 \\ \left(2\,t^3 - 3\,t^2 + 1\right) y1 + \left(3\,t^2 - 2\,t^3\right) y2 + \left(-2\,t^2 + t^3 + t\right) dy1 + \left(-t^2 + t^3\right) dy2 \\ \left(2\,t^3 - 3\,t^2 + 1\right) z1 + \left(3\,t^2 - 2\,t^3\right) z2 + \left(-2\,t^2 + t^3 + t\right) dz1 + \left(-t^2 + t^3\right) dz2 \end{bmatrix}$$

# Examples

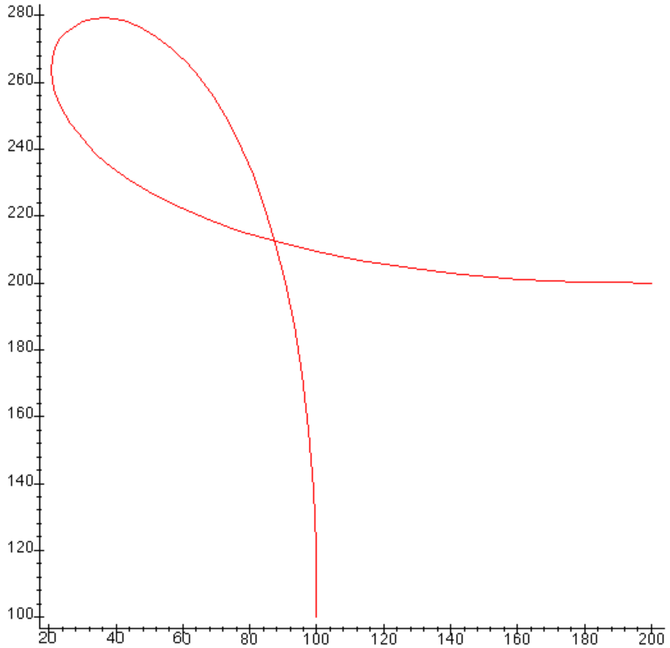


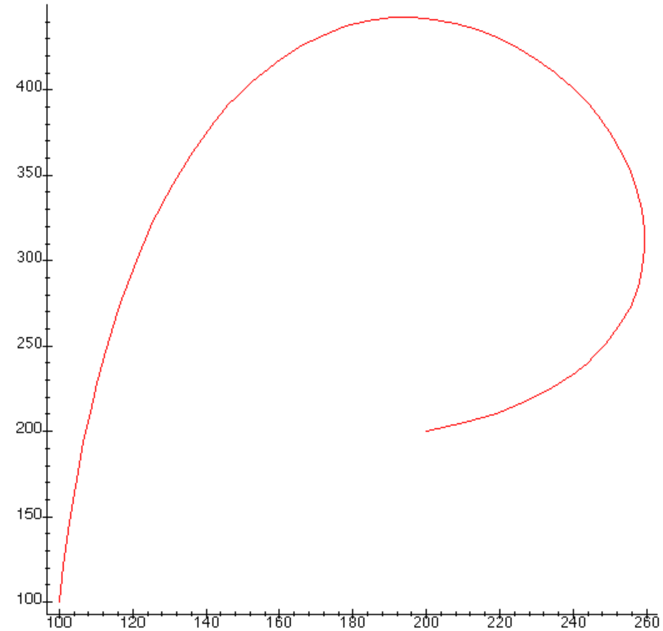




$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 10 & 10 & 0 \\ 10 & 10 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 0 & 200 & 0 \\ 200 & 0 & 0 \end{pmatrix}$$

# More Examples




$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 0 & 1000 & 0 \\ 1000 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 100 & 100 & 0 \\ 200 & 200 & 0 \\ 100 & 2000 & 0 \\ -500 & -200 & 0 \end{pmatrix}$$

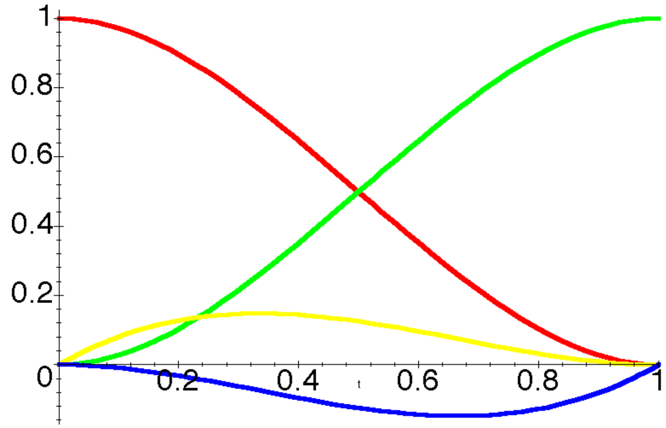
# Hermite Blending Functions

- Conceptual Realignment
  - Curves are weighted averages of points/vectors.
  - Blending functions specify the weighting.

$$Q = \begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} (2t^3 - 3t^2 + 1) + \begin{bmatrix} x2 \\ y2 \\ z2 \end{bmatrix} (-2t^3 + 3t^2) + \begin{bmatrix} dx1 \\ dy1 \\ dz1 \end{bmatrix} (t^3 - 2t^2 + t) + \begin{bmatrix} dx2 \\ dy2 \\ dz2 \end{bmatrix} (t^3 - t^2)$$

$$Bh_1 = 2t^3 - 3t^2 + 1$$

$$Bh_2 = -2t^3 + 3t^2$$




$$Bh_3 = t^3 - 2t^2 + t$$

$$Bh_4 = t^3 - t^2$$

# From Hermite to Bezier

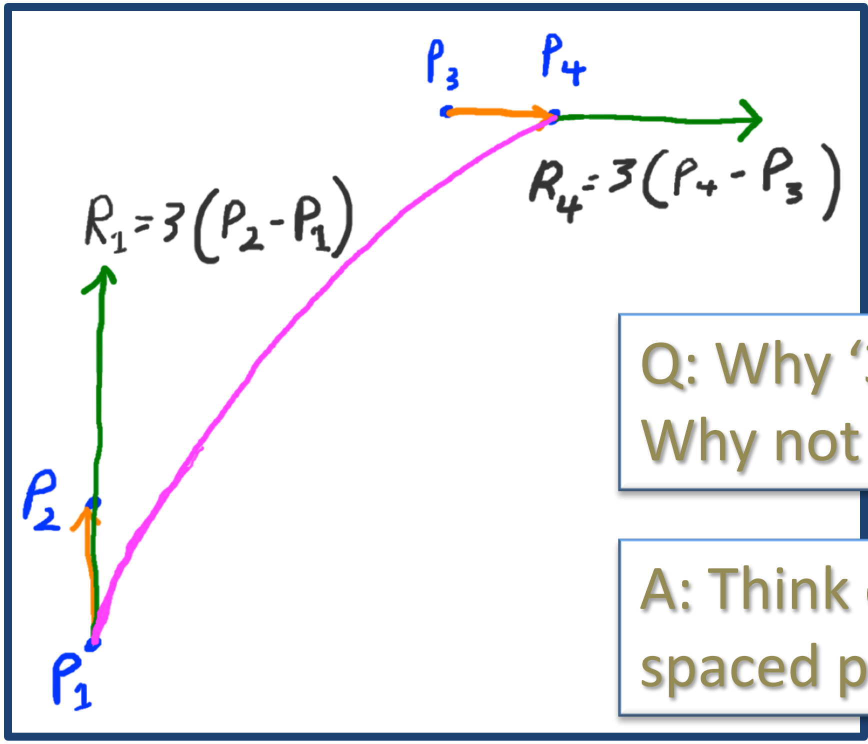
What's wrong with Hermite curves?

Nothing, unless you are using a point-and-click interface

Bezier curves are like Hermite curves, except that the user specifies four points ($p_1$, $p_2$, $p_{3,}$ $p_4$).
The curve goes through $p_1$ & $p_4$.
Points $p_2$ & $p_3$ specify the tangents at the endpoints.

# More Precisely....

$$R_1 = \frac{d}{dt}P_1 = 3(P_2 - P_1) \qquad R_4 = \frac{d}{dt}P_4 = 3(P_4 - P_3)$$




Tangents at start and end are now defined by intermediate points.

Q: Why '3'?
Why not $R_1=(P_2-P_1)$?

A: Think of 4 evenly spaced points in a line

# Hermite → Bezier

The Hermite geometry matrix is related to the Bezier geometry matrix by:

$$G_H = \begin{pmatrix} P1 \\ P4 \\ R1 \\ R4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} P1 \\ P2 \\ P3 \\ P4 \end{pmatrix}$$

$$= M_{HB} G_B$$

# Hermite $\rightarrow$ Bezier

For Hermite curves, $Q(t) = T\, M_H G_H$,
where $G_H = [P_1, P_4, R_1, R_4]^T$, $T = [t^3, t^2, t, 1]$

$$\text{and } M_H = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\text{So, } Q(t) = T\, \underbrace{M_H M_{HB}}\, G_B$$

# The Bezier Basis Matrix

$$Q(t) = T(M_H M_{HB}) G_B$$

$$M_B = M_H M_{HB} = \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$Q(t) = T M_B G_B$$

See page 364 to connect with description in our optional textbook.

# The Bezier Blending Functions

$$Q(t) = \begin{bmatrix} (-t^3 - 3t + 3t^2 + 1)\,x1 + (3t - 6t^2 + 3t^3)\,x2 + (3t^2 - 3t^3)\,x3 + t^3\,x4 \\ (-t^3 - 3t + 3t^2 + 1)\,y1 + (3t - 6t^2 + 3t^3)\,y2 + (3t^2 - 3t^3)\,y3 + t^3\,y4 \\ (-t^3 - 3t + 3t^2 + 1)\,z1 + (3t - 6t^2 + 3t^3)\,z2 + (3t^2 - 3t^3)\,z3 + t^3\,z4 \end{bmatrix}$$

$$\begin{aligned} Q(t) = {} & P_1(-t^3 + 3t^2 - 3t + 1) \\ & + P_2(3t^3 - 6t^2 + 3t) \\ & + P_3(-3t^3 + 3t^2) \\ & + P_4(t^3) \end{aligned}$$

# Add them up.

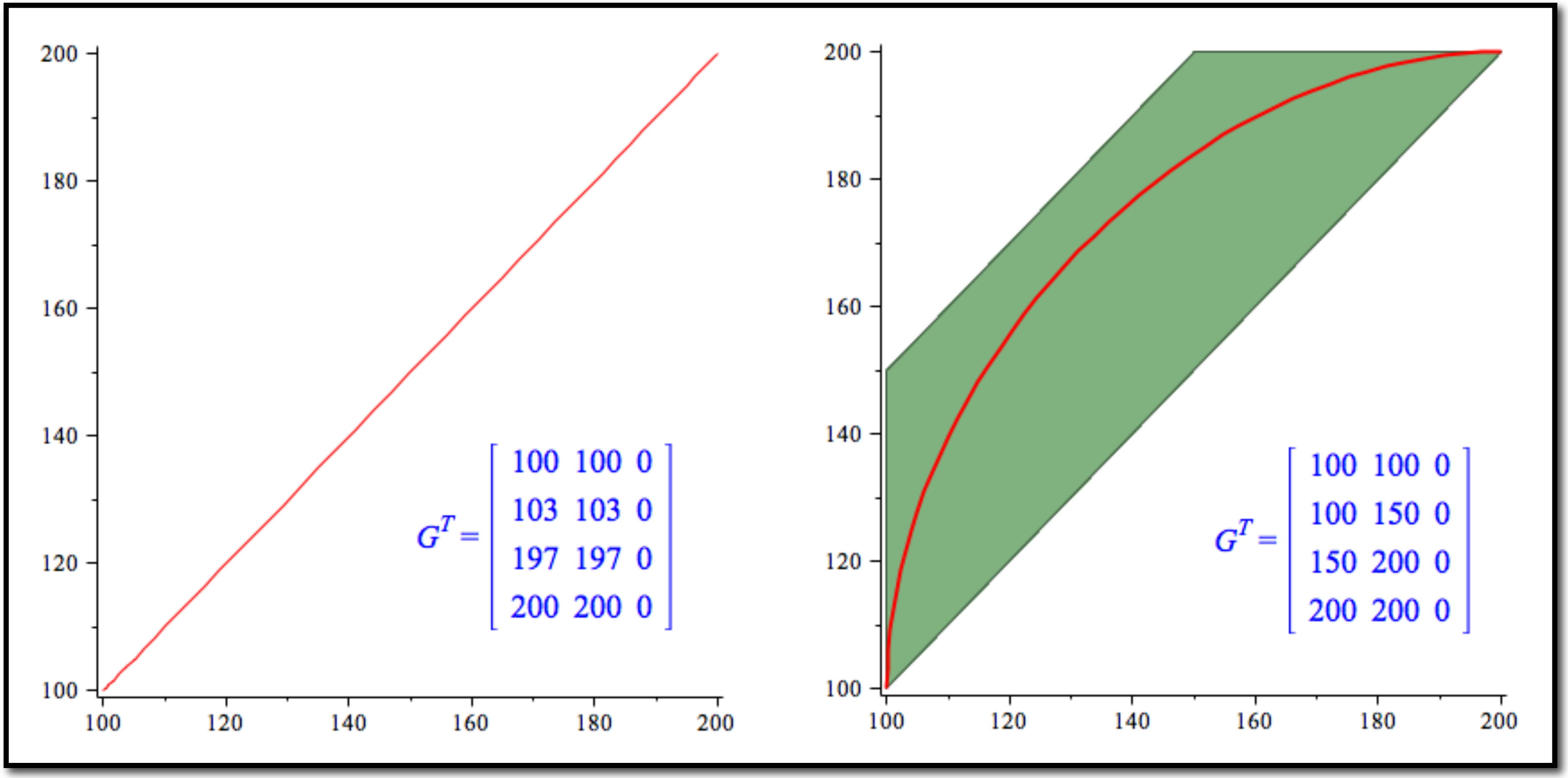
$$
\begin{aligned}
&\;\;\;(-t^3 + 3t^2 - 3t + 1) \\
&+ (3t^3 - 6t^2 + 3t) \\
&+ (-3t^3 + 3t^2) \\
&+ (t^3) \\
\hline
&\;\;\;0t^3 + 0t^2 + 0t + 1
\end{aligned}
$$

# Stay within the Convex Hull

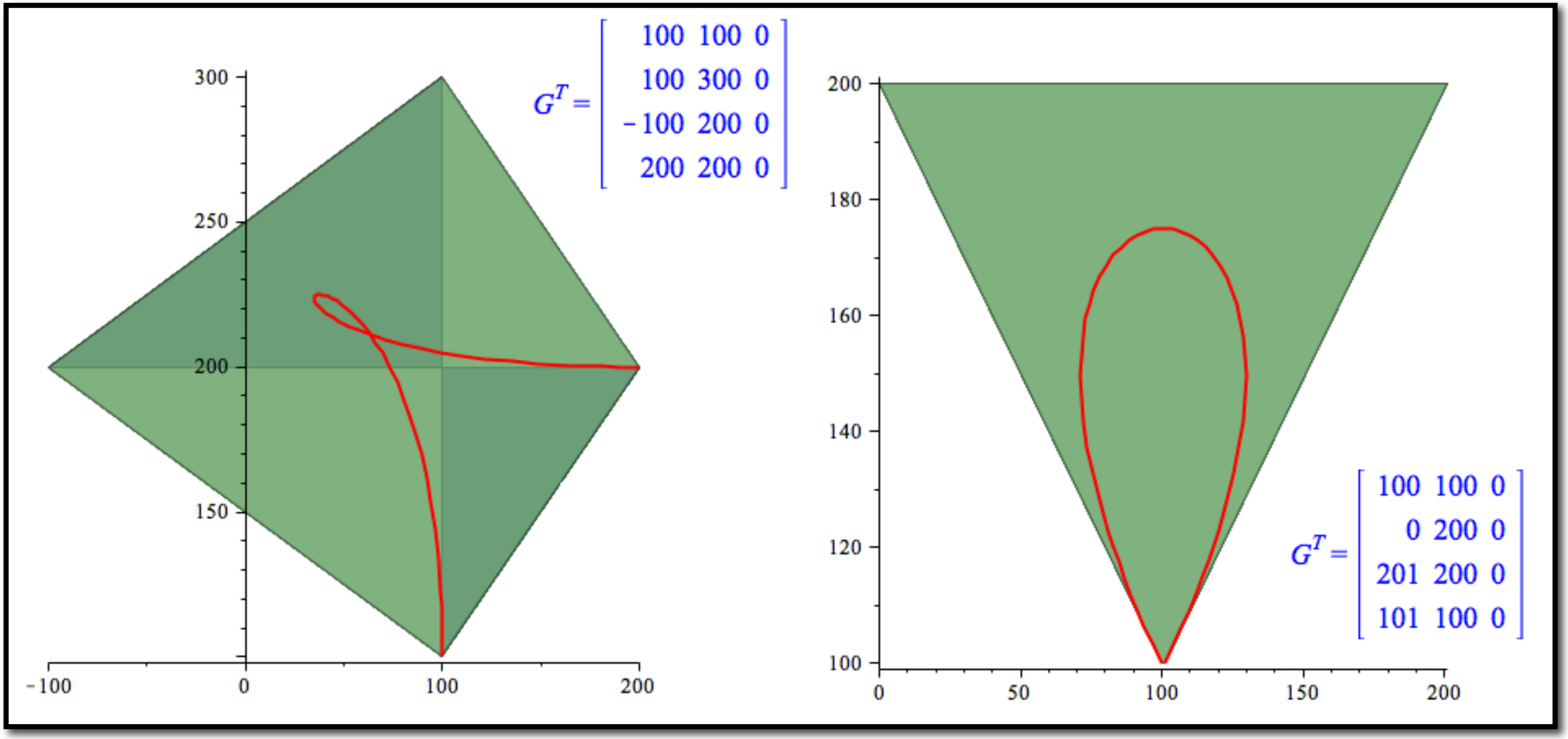
If you graph the four Bezier blending functions for t=0 to t=1, you find that they are always positive and always sum to one.

Therefore, *the Bezier curve stays within the convex hull defined by* $P_1$, $P_2$, $P_3$ & $P_4$.

# Examples 1.


200
180
160
140
120
100
100
120
140
160
180
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 103 & 103 & 0 \\ 197 & 197 & 0 \\ 200 & 200 & 0 \end{bmatrix}$
200
180
160
140
120
100
100
120
140
160
180
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 100 & 150 & 0 \\ 150 & 200 & 0 \\ 200 & 200 & 0 \end{bmatrix}$

# Examples 2.

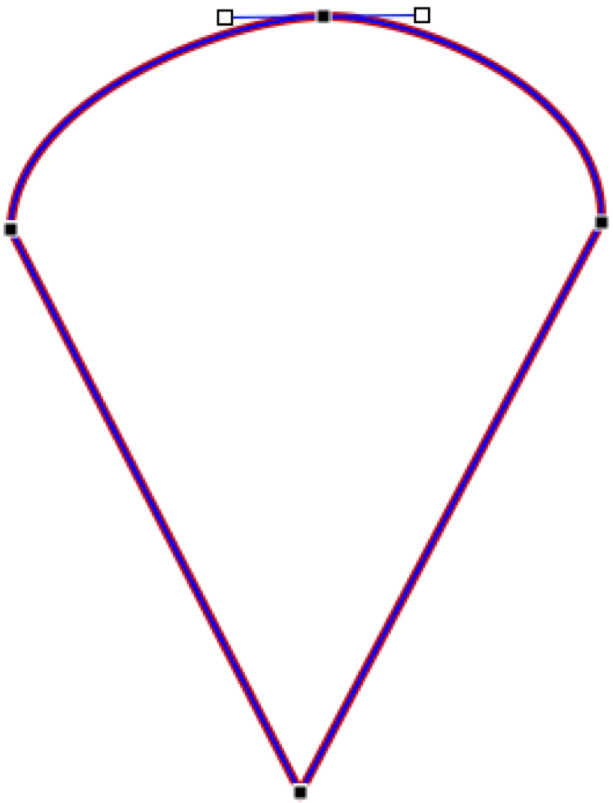

$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 100 & 300 & 0 \\ -100 & 200 & 0 \\ 200 & 200 & 0 \end{bmatrix}$
300
250
200
150
-100
0
100
200
200
180
160
140
120
100
0
50
100
150
200
$G^T = \begin{bmatrix} 100 & 100 & 0 \\ 0 & 200 & 0 \\ 201 & 200 & 0 \\ 101 & 100 & 0 \end{bmatrix}$

# Example 3D

```
P1 = (  0,   0,   0)
P2 = (100,   0,   0)
P3 = (  0, 100, 100)
P4 = (  0,   0, 100)
```

# Bezier Curves are Common

- You have probably already used them.
- For example, in PowerPoint
  - Build a shape
  - Then select edit points
  - Notice the control ‘wings’
- Enhancements
  - Ways to introduce constraints
    - Smooth Point
    - Straight Point
    - Corner Point



# Stepping Back

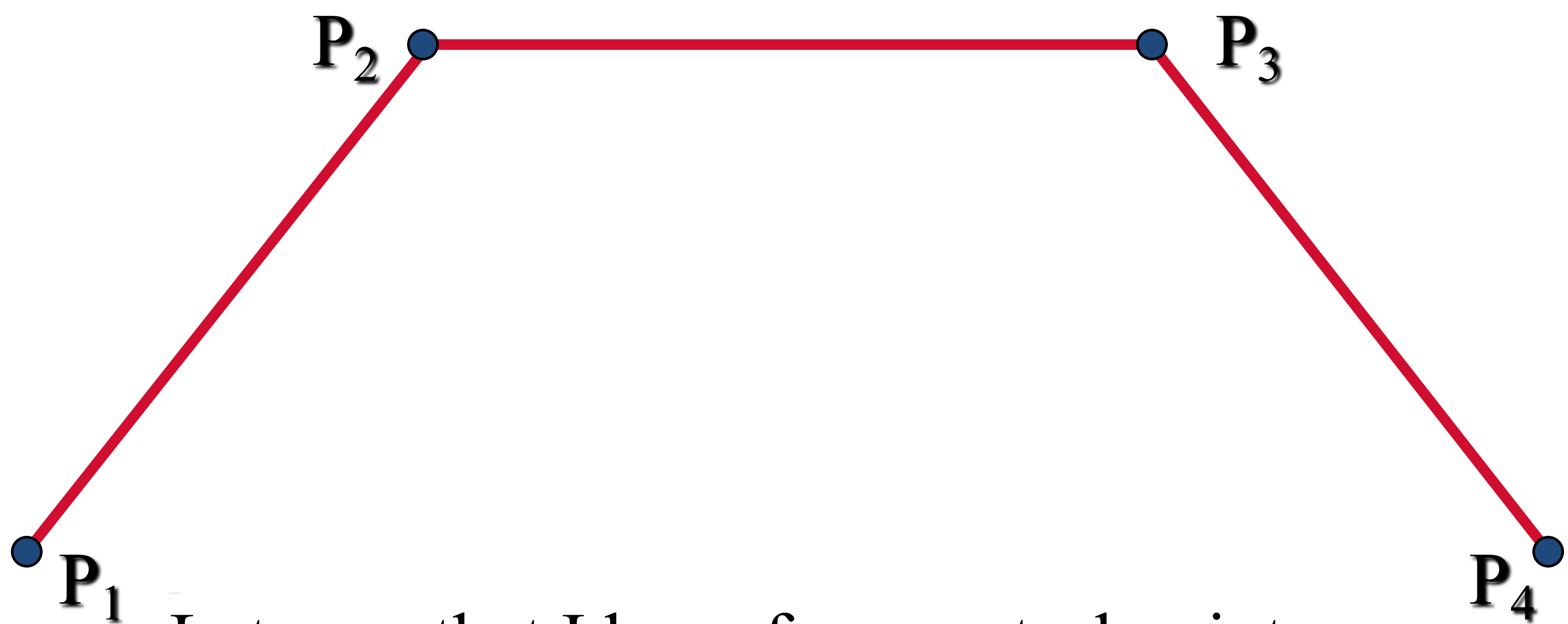
What should you be learning? Should you memorize $M_H$ and $M_B$? No! That's what reference books are for.

You should know what $G_H$ and $G_B$ are. You should know how to derive $M_H$ from the parametric form of the cubic equations. You should know how to derive $M_B$ from $M_H$. If you understand these concepts, you can look up or rederive the matrices as necessary.

# de Casteljau Curves

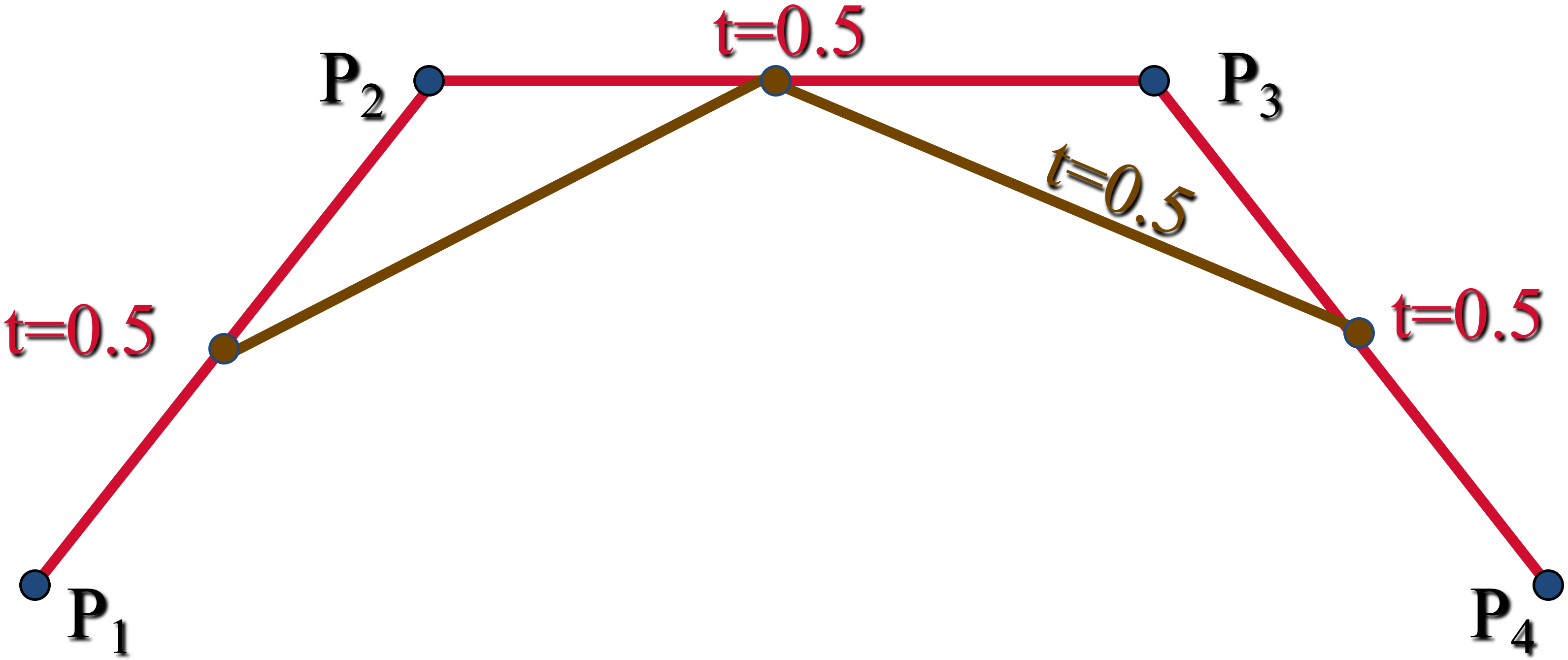
***Now for something completely different.***

There is another way to motivate curves.



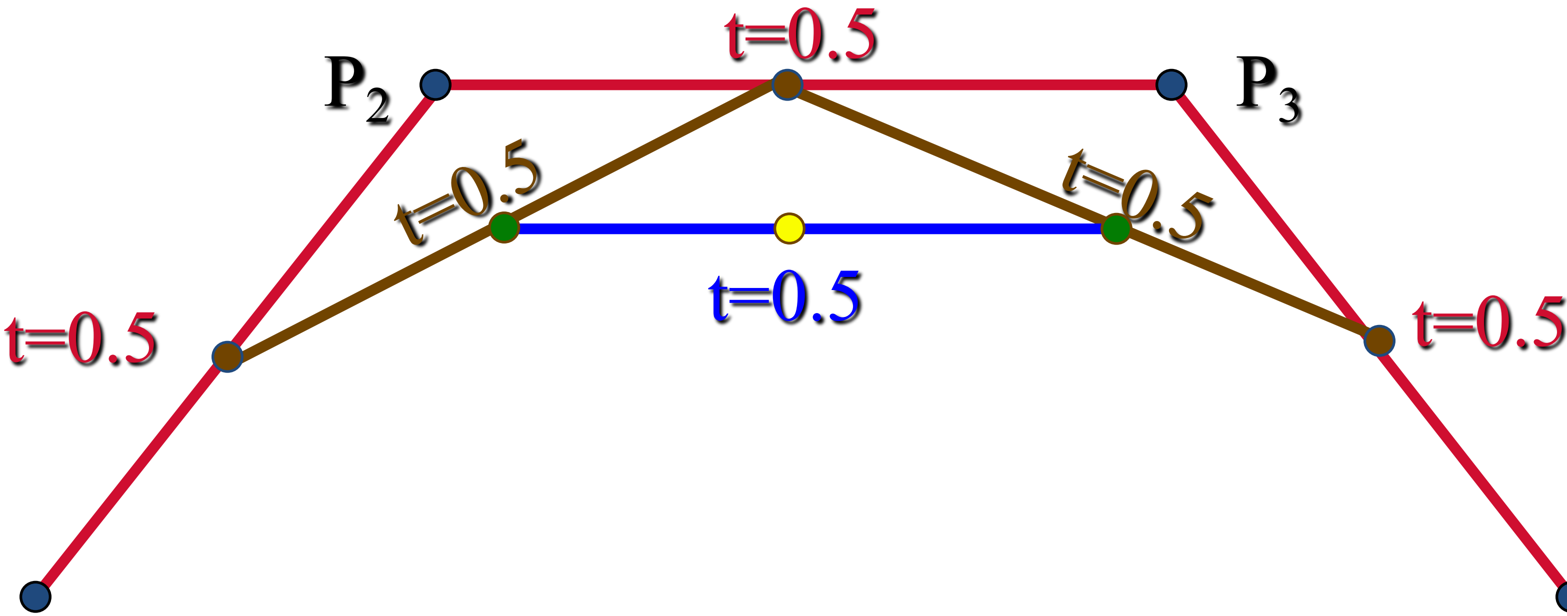

Lets say that I have four control points

To find the midpoint of the curve corresponding to those control points:




Connect the point between $P_1$ and $P_2$ where t = 0.5 with the point between $P_3$ & $P_2$ where t = 0.5;
Do the same with the $P_2$-$P_3$ & $P_3$-$P_4$ lines

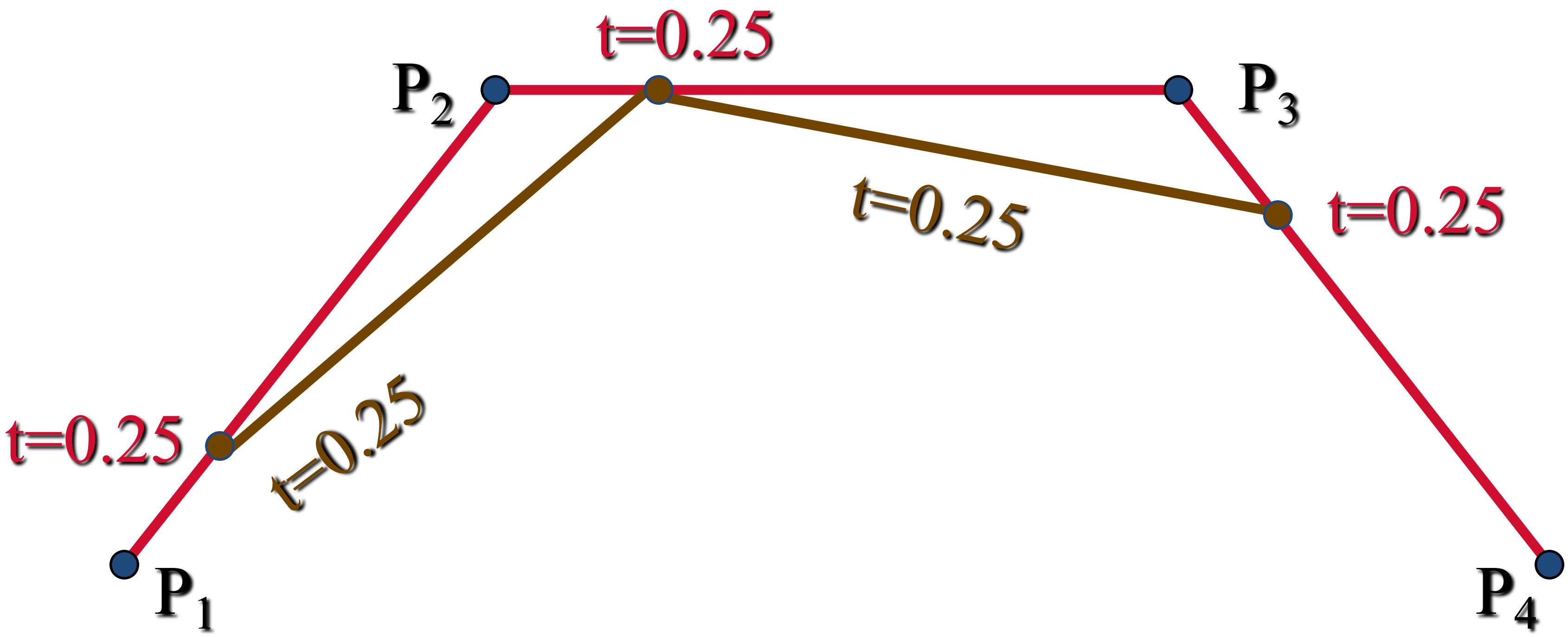
# Now, connect these two lines at their t = 0.5 points



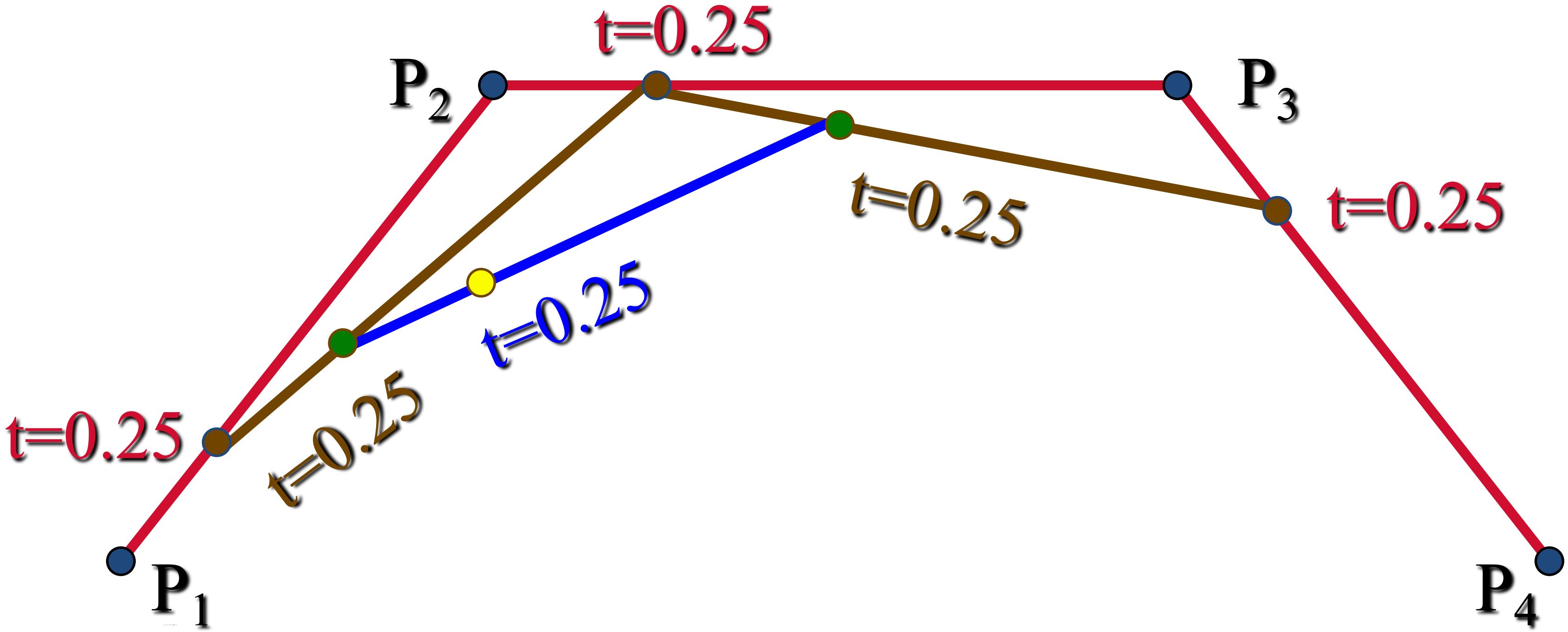

The t = 0.5 point on the resulting segment is the midpoint (t = 0.5 point) of some type of curve which is made up of weighted averages of the control points (*we'll soon see what kind of curve*)

We can extend this idea to any t value.

To compute the t = 0.25 point, connect the 0.25 points of the original lines...



Now, connect these lines at their t = 0.25 locations, and find the point where t = 0.25 of the resulting line




In this way, you can compute the 3rd-order curve for any value of t

# Algebraic Definition

- The equations of the three original lines are:

$$A_1(t) = (1 - t)\,P_1 + t\,P_2$$

$$A_2(t) = (1 - t)\,P_2 + t\,P_3$$

$$A_3(t) = (1 - t)\,P_3 + t\,P_4$$

- The equations of the next two joining lines are:

$$B_1(t) = (1 - t)\,A_1 + t\,A_2$$

$$B_2(t) = (1 - t)\,A_2 + t\,A_3$$

- Finally, the line between the two joining lines is:

$$C_1(t) = (1 - t)\,B_1 + t\,B_2$$

# Begin Substitutions

- Substitute equations for $A_1$ and $A_2$ into $B_1$

$$B_1(t) = (1-t)\,((1-t)\,P_1 + t\,P_2) + t\,((1-t)\,P_2 + t\,P_3)$$
$$= P_1 - 2\,t\,P_1 + t^2\,P_1 + 2\,t\,P_2 - 2\,t^2\,P_2 + t^2\,P_3$$
$$= (t^2 + 1 - 2\,t)\,P_1 + (-2\,t^2 + 2\,t)\,P_2 + t^2\,P_3$$

***Factoring the result***

$$B_1(t) = (t-1)^2\,P_1 - 2\,t\,(t-1)\,P_2 + t^2\,P_3$$

- Likewise, substitute $A_2$ and $A_3$ into $B_2$

$$B_2(t) = (t-1)^2\,P_2 - 2\,t\,(t-1)\,P_3 + t^2\,P_4$$

# Resulting Third Order Curve

Substitute equations for $B_1$ and $B_2$ into $C_1$

$$C_1(t) = (1-t)\,((t-1)^2 P_1 - 2\,t\,(t-1)\,P_2 + t^2 P_3) + t\,((t-1)^2 P_2 - 2\,t\,(t-1)\,P_3 + t^2 P_4)$$

$$= (-3\,t - t^3 + 3\,t^2 + 1)\,P_1 + (3\,t^3 - 6\,t^2 + 3\,t)\,P_2 + (-3\,t^3 + 3\,t^2)\,P_3 + t^3 P_4$$

***And After Factoring***

$$C_1(t) = (1-t)^3 P_1 + 3\,t\,(t-1)^2 P_2 - 3\,t^2 (t-1)\,P_3 + t^3 P_4$$

# Bezier = de Casteljau

But those last four functions are exactly the Bezier blending functions!

The recursive line intersection algorithm can therefore be used to gain intuition about the behavior of Bezier functions

***Not something completely different afterall.***