Optimization Problems and Local Search

Russell and Norvig 4.1
Optimization Problems

- Previously: systematic exploration of search space.
  - Path to goal is the solution
- For some problems path is irrelevant.
  - Example: 8-queens
Stated as an optimization problem:
- State space: a board with 8 queens on it
- Objective/cost function: Number of pairs of queens that are attacking each other (quality of the state).
The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

An optimal TSP tour through Germany’s 15 largest cities (one out of 14!/2)

13,509 cities and towns in the US that have more than 500 residents

http://www.tsp.gatech.edu/
The Traveling Salesman Problem (TSP)

**TSP:** Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.
Local Search

- Keep a current state, try to improve it by “locally” exploring the space of solutions
- Improve state by moving a queen to a position where fewer queens attack each other (neighboring state)
- Neighbors: move a queen in its column
Greedy local search

- Problem: can get stuck in a local minimum (happens 86% of the time for the 8-queens problem).
Local minima vs. local maxima

- Local search: find a local maximum or minimum of an objective function (cost function).

- Local minima of a function $f(n)$ are the same as the maxima of $-f(n)$. Therefore, if we know how to solve one, we can solve the other.
Hill-climbing

Try all neighbors and keep moves that improve the objective function the most

- Objective function
- Global maximum
- Local maximum
- “Flat” local maximum
- Shoulder
- Plateau
- Current state
- State space

Diagram showing the objective function over the state space with peaks representing maxima and plateaus.
Hill-climbing

function HILL-CLIMBING( problem) return a state that is a local maximum

current ← MAKE-NODE(problem.INITIAL-STATE)
loop do
  neighbor ← a highest valued successor of current
  if neighbor.VALUE ≤ current.VALUE then return current.STATE
  current ← neighbor

This flavor of hill-climbing is known as steepest ascent
(steepest descent when the objective is minimization)

Finds local optimum 86% of the time for 8 queens problem.
Hill-climbing

Try all neighbors and keep moves that improve the objective function the most.

What makes plateaus a challenge for hill-climbing?
Formulating a problem as a local search problem

What you need to decide on:

- The possible states and their representation
- Choice of initial state
- Choice of neighborhood of a state
  - The neighborhood should be rich enough such that you don’t get stuck in bad local optima
  - It should be small enough so that you can efficiently search the neighbors for the best local move
Solving TSP

Need to design a neighborhood that yields valid tours

A 2-opt move:
3-opt

- Choose three edges from tour
- Remove them, and combine the three parts to a tour in the cheapest way to link them
Solving TSP (cont.)

- 3-opt moves lead to better local minima than 2-opt moves.
- The Lin-Kernighan algorithm (1973): a $\lambda$-opt move - constructs a successor that changes $\lambda$ cities in a tour
- Often finds optimal solutions.
Variations of hill climbing

- Steepest ascent: choose the neighbor with the largest increase in objective function.
- Stochastic hill-climbing
  - Random selection among the uphill moves.
  - The selection probability can vary with the steepness of the uphill move.
- First-choice hill-climbing
  - Stochastic hill climbing, generating successors randomly until a better one is found.
- Random-restart hill-climbing
  - Choose best among several hill-climbing runs, each from a different random initial state.
Suppose that the probability of failure in a single try is $P_f$

The probability of failure in $k$ trials:

$$P_f(k\ \text{trials}) = (P_f)^k$$

$$P_s(k\ \text{trials}) = 1 - P_f(k\ \text{trials}) = 1 - (P_f)^k$$

The probability of success can be made arbitrarily close to 1 by increasing $k$.

Example: For the eight queens problem

$$P_s(100\ \text{trials}) = 0.9999997$$
Hill climbing for NP-complete problems

- NP-complete problems can have an exponential number of local minima.

- But:
  - Most instances might be easy to solve
  - Even if we can’t find the optimal solution, a reasonably good local maximum can often be found after a small number of restarts.