Introduction to Fourier Analysis

CS 510

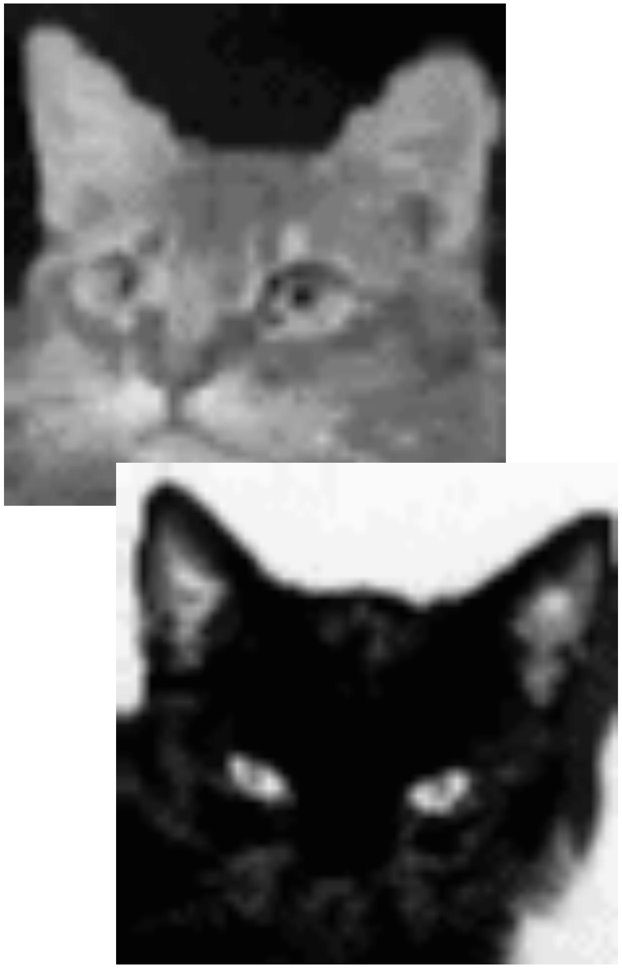
Lecture #6

February 6th, 2013

Colorado State University

Details are on the assignments page… you have until Monday

Programming Assignment #1



Source (Target) Images

Templates

Left Eye

Right Eye

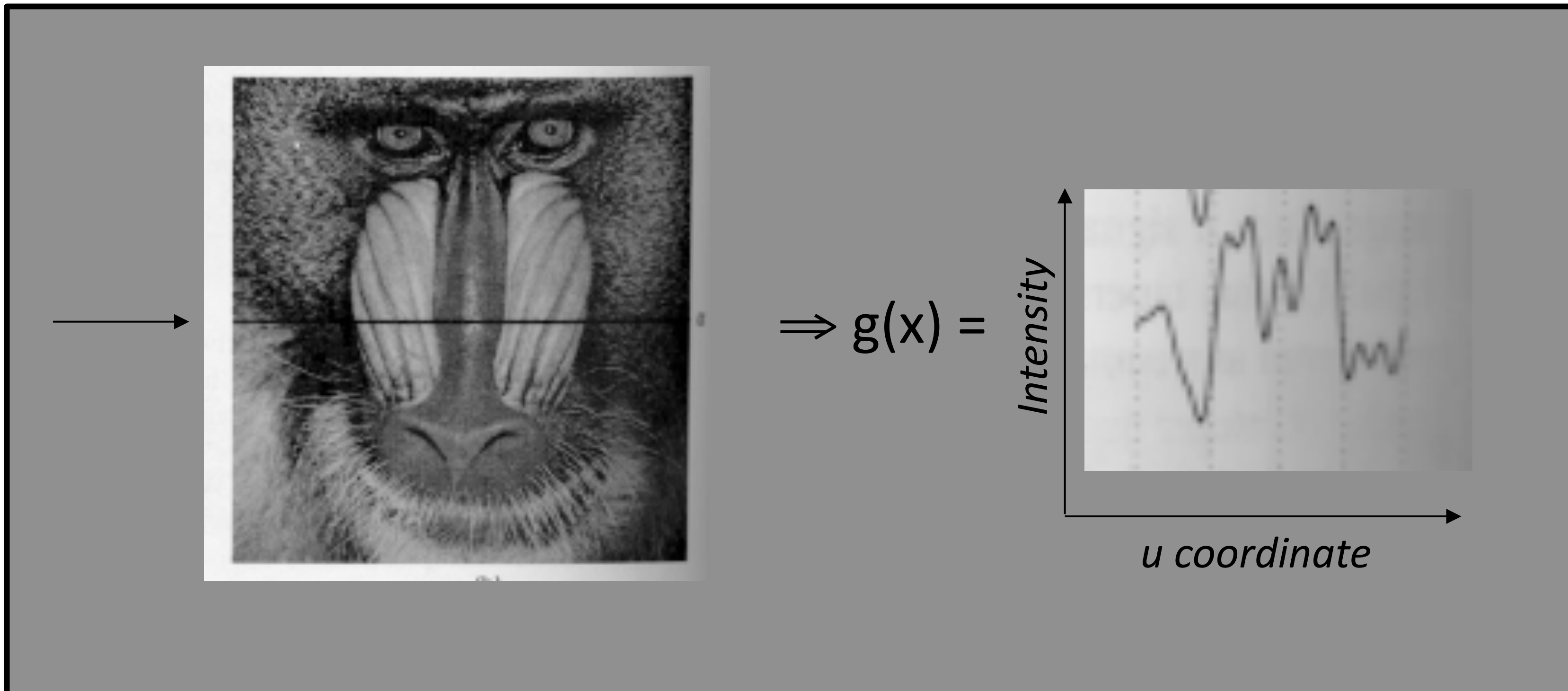
Left Ear

Nose

Motivation for Fourier Analysis

- We will now briefly detour into Fourier Analysis
- Why?
  - It will make template matching faster
  - It will help us build better templates
  - It will improve our intuitive understanding of image space
- Other benefits we will not address
  - Understand image aliasing, Nyquist rates, etc.
  - Understand JPG compression

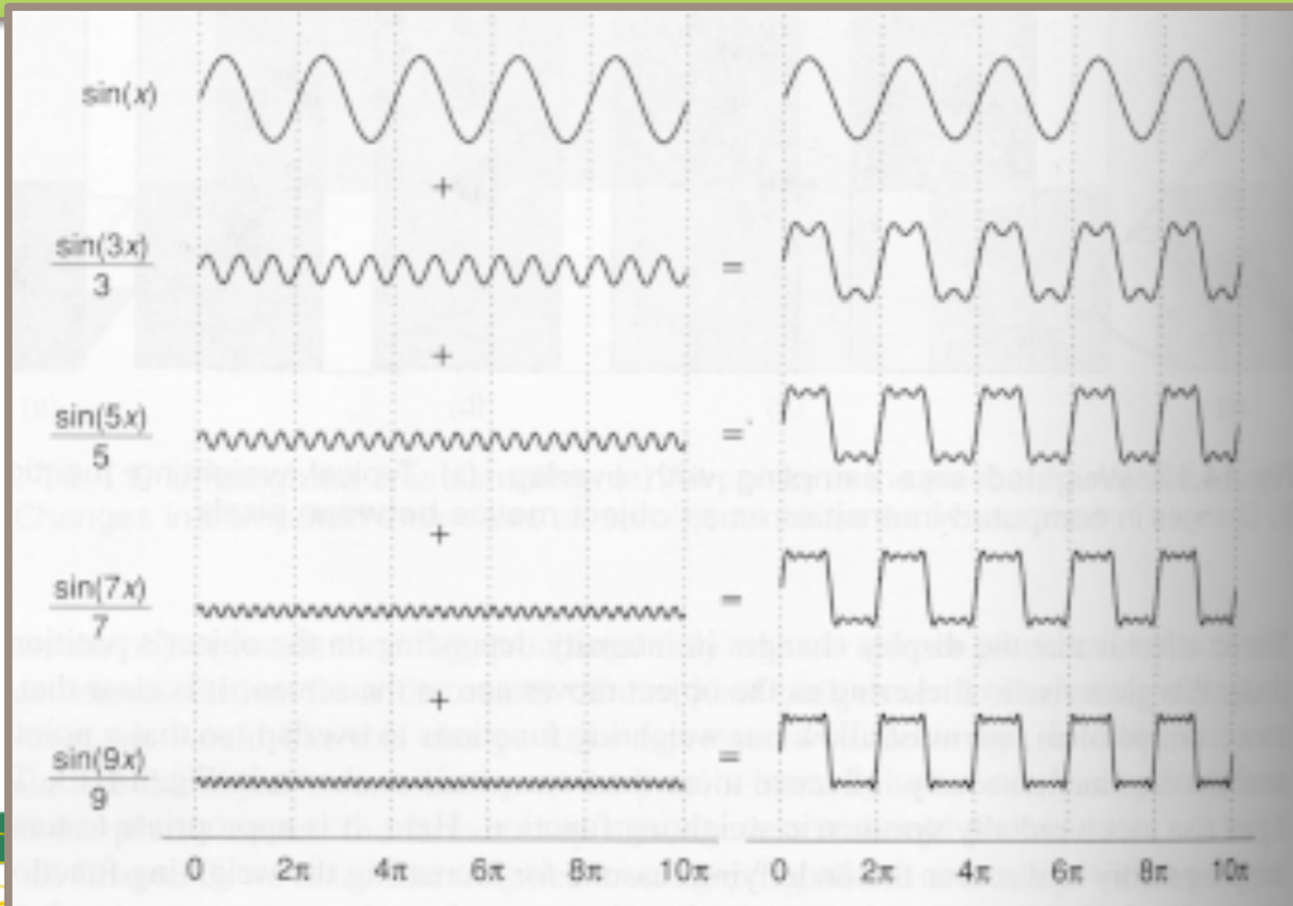
Image Data as a Function



Graphic from *"Computer Graphics:Principles and Practice"* *by Foley, van Dam, Feiner & Hughes.*

Sampling

Any repeating pattern can be constructed from an (infinite number of) sine waves.



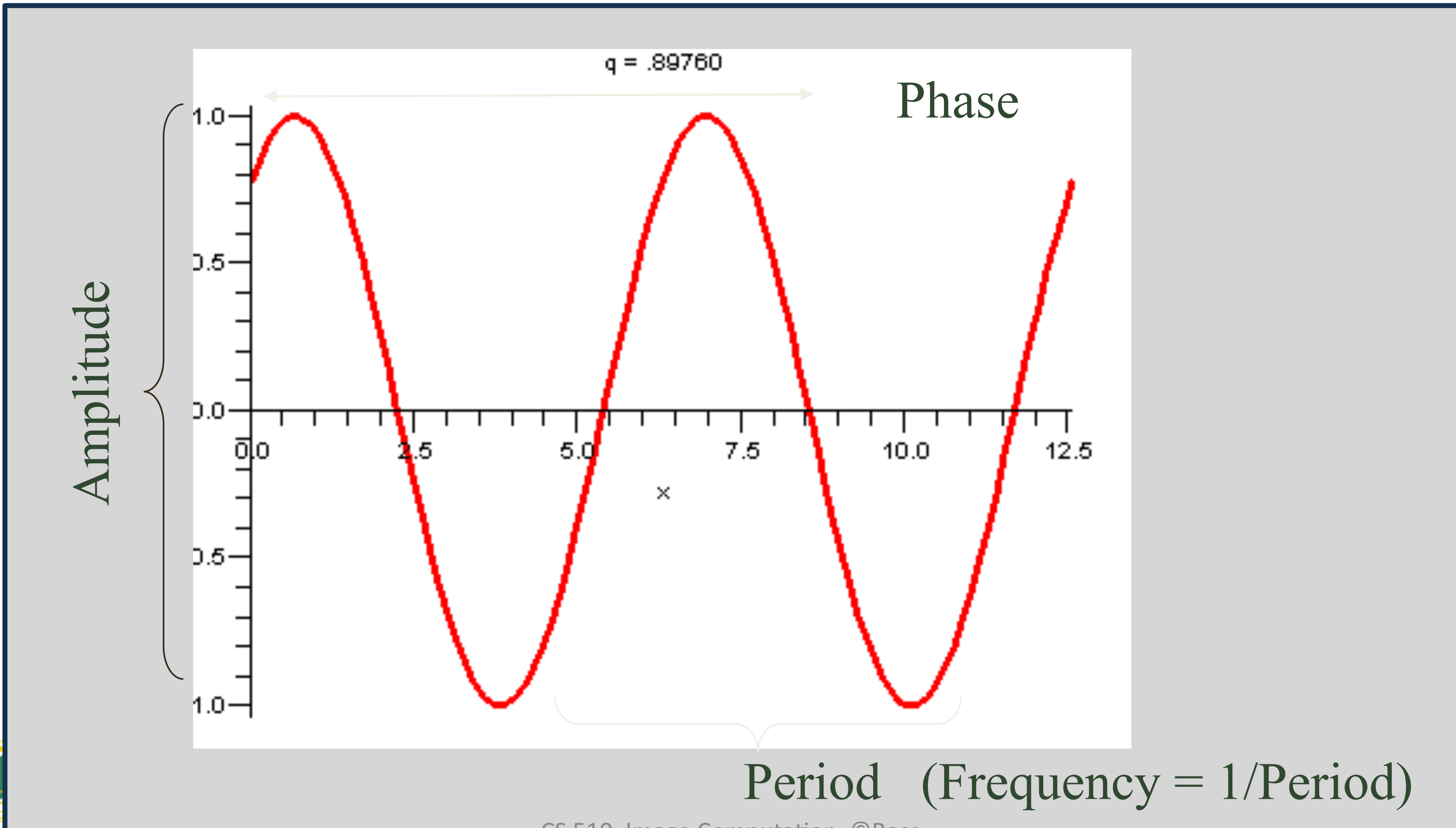
Graphic from *"Computer Graphics: Principles and Practice"* by Foley, van Dam, Feiner & Hughes.

Fourier Analysis ≠ Magic

- Many textbooks make is obscure, but…
- We are just rewriting a function f(x) over a finite range.
  - In effect, we pretend it repeats.
  - And reconstruct it as a sum of sine waves …
- For each sine wave, we specify:
  - A frequency
  - An amplitude
  - A phase

The Sine Wave

This may be a review from high school, but…



The Sine Wave (II)

$$g(x) = a\cos(fx + \phi)$$

Frequency

Amplitude

Phase

Simplifying Phase

- Phase describes where the cycle crosses the x axis:
  - If it crosses at 0 and -π, it's a sine wave.
  - If it crosses at $\pi/2$ and - $\pi/2$, it a cosine wave.
  - In general, if it crosses at ϕ and $\phi + \pi$ radians, it has phase $\phi - \pi/2$ (i.e., cosine, not sine)
    - $\phi = 0 \Rightarrow$ cosine wave
    - $\phi = \pi \Rightarrow$ sine wave
- A wave with phase ϕ can be expressed as:

$$\cos(x+\phi) = \alpha\cos(x) + \beta\sin(x)$$

Phase (II)

Where:

$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\cos(\theta + \phi) = \sqrt{\alpha^2 \cos^2(\theta) + \beta^2 \sin^2(\theta)}$$

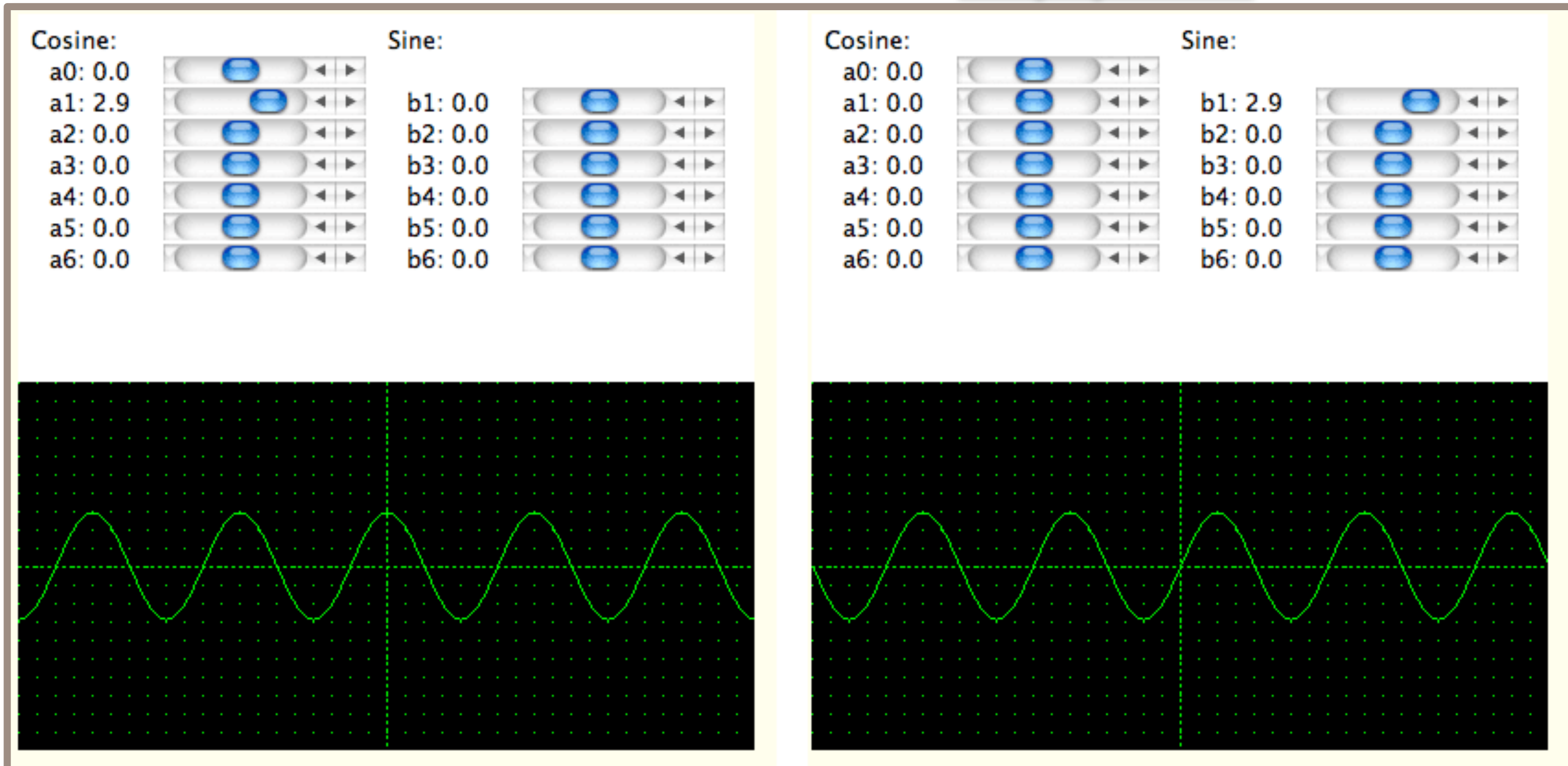
$(\theta + \phi)$ still indicates that the cosine curve has been shifted by ϕ degrees.

Therefore…

$$g(x) = a_1 \cos(f_1 x) + b_1 \sin(f_1 x) + a_2 \cos(f_2 x) + b_2 \sin(f_2 x) + a_3 \cos(f_3 x) + b_3 \sin(f_3 x) + \cdots$$

- Remaining problems
  - Now has twice as many terms (2 per frequency)
  - Must specify amplitude and frequency for each

Visualization - Nice Applet



Applet by Manfred Thole,
www.nst.ing.tu-bs.de/schaukasten/fourier/en_idx.html

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Now, Per Frequency

- For any given frequency f_1 , we must calculate the amplitude of the cosine and sine functions
- G(x) is infinite (x can be anything), so we have to fit the cosines and sines to all the data:

$$a_1 = \int_{-\infty}^{+\infty} f(x)\cos(f_1 x)\,dx \qquad b_1 = \int_{-\infty}^{+\infty} f(x)\sin(f_1 x)\,dx$$

Representing Frequencies

- Our function g(x) is infinitely repeating
- Assume, WLOG, the unit of repetition is 1
  - So g(0+a) = g(1+x) = g(2+a)…
- Only cosines & sines with periods that are integers make sense
  - Otherwise is wouldn't repeat
- The cosine whose period is 1 is written as

$$\cos(2\pi x)$$

Frequencies (cont).

- In general the frequency that repeats u times over the interval is written as

$$\cos\left(2\pi u x\right)$$

- Where u is any integer

So, to rewrite g(x)

- For every integer frequency u = 1, 2, 3, …

$$
\begin{aligned}
g(x) = {} & a_1 \cos(2\pi x) + b_1 \sin(2\pi x) \\
& + a_2 \cos(2\pi 2x) + b_2 \sin(2\pi 2x) \\
& + a_3 \cos(2\pi 3x) + b_3 \sin(2\pi 3x) \\
& + a_4 \cos(2\pi 4x) + b_4 \sin(2\pi 4x) \\
& + \ldots
\end{aligned}
$$

$$a_u = \int_{-\infty}^{\infty} g(x) \cos(2\pi ux)\, dx \quad b_u = \int_{-\infty}^{\infty} g(x) \sin(2\pi ux)\, dx$$

Fourier Transform

- Formally, the Fourier transform in 1D is:

$$F(u) = \int_{-\infty}^{+\infty} f(x)\left[\cos 2\pi ux - i\sin 2\pi ux\right]dx$$

Where:

u is an integer in the range from 0 to ∞

–i is used to create a 2D vector space

$F(u) = a_u + ib_u$

The DC component

- What happens when u = 0?
  - cos(0) = 1, sin(0) = 0
- So

$$F(u=0) = \int_{-\infty}^{+\infty} f(x)\left[\cos 2\pi ux - i\sin 2\pi ux\right]dx$$

$$= \int_{-\infty}^{+\infty} f(x)dx$$

  - This is the average value (or “DC component”) of the function. For images, it is largely a function of lighting.

Inverse Fourier Transform

- What if I have F(u) for all u, and I want to recreate the original function g(x)?

- Well, sum it up for every u:

$$f(x) = \int_{-\infty}^{+\infty} F(u)\left[\cos(2\pi ux) + i\sin(2\pi ux)\right]du$$

Discrete Fourier Transform

- Problem: an image is not an analogue signal that we can integrate.
- Therefore for $0 \leq x < N$ and $0 \leq u < N/2$:

$$F(u) = \sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

And the discrete inverse transform is:

$$f(x) = \frac{1}{N}\sum_{x=0}^{N-1} F(u)\left[\cos\left(\frac{2\pi ux}{N}\right) + i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

Discrete vs. Continuous

- Summation replaces integration
- Division by N (the number of discrete samples) makes the unit of repetition 1.
- For any signal (continuous or discrete)
  - G(x) is called the spatial domain
  - F(u) is called the frequency domain

Spatial vs Frequency

- Spatial domain representation size?
  - Given N samples, it is size N
- Frequency domain representation size?
  - A total of N/2 frequencies
  - A complex number (2 values) per frequency
- The DFT is invertible, so the two representations are equivalent:
  - Exact same information and same size
  - The FFT is O(n log n)