Introduction to Fourier Analysis – Part 2

CS 510
Lecture #6
February 9, 2020
In the extreme, a square wave

Graphic from http://www.mechatronics.colostate.edu/figures/4-4.jpg
Fourier Transform

• Formally, the Fourier transform in 1D is:

\[ F(u) = \int_{-\infty}^{+\infty} f(x)[\cos 2\pi ux - i \sin 2\pi ux] \, dx \]

Where:

- \( u \) is an integer in the range from 0 to \( \infty \)
- \(-i\) is used to create a 2D vector space
- \( F(u) = a_u + ib_u \)
Inverse Fourier Transform

• What if I have $F(u)$ for all $u$, and I want to recreate the original function $f(x)$?

• Well, sum it up for every $u$:

$$f(x) = \int_{-\infty}^{+\infty} F(u)[\cos(2\pi ux) + i \sin(2\pi ux)] \, du$$
Discrete Fourier Transform

- Problem: an image is not an analogue signal that we can integrate.
- Therefore for \(0 \leq x < N\) and \(0 \leq u < N/2\):

\[
F(u) = \sum_{x=0}^{N-1} f(x) \left[ \cos\left(\frac{2\pi ux}{N}\right) - i \sin\left(\frac{2\pi ux}{N}\right) \right]
\]

And the discrete inverse transform is:

\[
f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) \left[ \cos\left(\frac{2\pi ux}{N}\right) + i \sin\left(\frac{2\pi ux}{N}\right) \right]
\]
Discrete vs. Continuous

• Summation replaces integration

• Division by N (the number of discrete samples) makes the unit of repetition 1.

• For any signal (continuous or discrete)
  – f(x) is called the spatial domain
  – F(u) is called the frequency domain
Spatial vs. Frequency

- **Spatial domain representation size?**
  - Given N samples, it is size N

- **Frequency domain representation size?**
  - A total of N/2 frequencies
    - Often plotted from –N/2 to N/2, but half are redundant
  - A complex number (2 values) per frequency

- **The DFT is invertible, so the two representations are equivalent:**
  - Exact same information and same size
  - The DFFT is $O(n \log n)$
Key Intuition: invertibility
An important intuition: “trimming” creates artifacts

Signal in spatial domain

Signal in frequency domain (real in blue, imaginary in green)

Reconstructed signal in spatial domain

http://madebyevan.com/dft/
An intuition for later...

Any frequency of the discrete Fourier transform

\[
F(u) = \sum_{x=0}^{N-1} f(x) \left[ \cos\left(\frac{2\pi ux}{N}\right) - i\sin\left(\frac{2\pi ux}{N}\right) \right]
\]

Can be rewritten as a complex dot product:

\[
F(u) = \left[ \cos\left(\frac{2\pi u0}{N}\right) - i\sin\left(\frac{2\pi u0}{N}\right), \ldots \right] \cdot \begin{bmatrix}
\vdots \\
f(0) \\
\vdots \\
f(N - 1)
\end{bmatrix}
\]

So the Fourier transform is linear.
An intuition ... (cont.)

The full transform is a matrix equation

Each circle represents a complex number; $x$ is cosine, $y$ is sine

An intuition ... (part 3)

- This matrix has special properties
  - Every row is orthogonal to every other
  - Every row has length $\sqrt{N}$
  - It is a rotation and a scale of image space
  - The inverse Fourier counter-rotates and counter scales
2D Fourier Transform

- So far, we have looked only at 1D signals.
- For 2D signals, the continuous generalization is:
  \[ F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \left[ \cos(2\pi(ux + vy)) - i \sin(2\pi(ux + vy)) \right] \]
  
- Note that frequencies are now two-dimensional:
  - \( u = \text{freq in } x \), \( v = \text{freq in } y \).
- Every frequency \((u,v)\) has a real and an imaginary component.
2D sine waves

• This looks like you’d expect in 2D

➢ Note that the frequencies don’t have to be equal in the two dimensions.

2D Discrete Fourier Transform

\[ F(u, v) = \sum_{x=-N/2}^{N/2} \sum_{y=-N/2}^{N/2} f(x, y) \left[ \cos\left(\frac{2\pi}{N} (ux + vy)\right) - i \sin\left(\frac{2\pi}{N} (ux + vy)\right) \right] \]

- What happened to the bounds on \( x \) & \( y \)?
- How big is the discrete 2D frequency space representation?
2D Frequency Space

• Remember that:
  • Cosine is an even function: \( \cos(x) = \cos(-x) \)
  • Sine is an odd function: \( \sin(x) = -\sin(-x) \)

• So
  • \( F(u,v) = a+ib \Rightarrow F(-u, -v) = a-ib \)

• And
  • \( F(-u,v) = a+ib \Rightarrow F(u, -v) = a-ib \)

• But
  • \( F(u,v) = a+ib \Rightarrow F(-u, v) = ??? \)
2D Frequency Space (cont)

• Size of 2D Frequency representation:
  – One dimension must vary from $-N/2$ to $N/2$, while the other varies from 0 to $N/2$
    • Doesn’t matter which is which
  – $N \times (N/2) \times 2$ values per frequency = $N^2$
  – Same as the source spatial representation
Showing Frequency Space

• To display a frequency space:
  – We plot it from $-N/2$ to $N/2$ in both dimensions
  – The result is symmetric about the origin (and therefore redundant)
  – We can’t plot a complex number, so we show the magnitude at every pixel $\sqrt{a^2 + b^2}$
    • Thus discarding the phase information
    • Phase plots are also possible ($\tan^{-1}(b/a)$)
Transform as Linear Operation

The full transform is a matrix equation

\[
\begin{bmatrix}
F1 \\
F2 \\
F3 \\
F4 \\
F5 \\
F6 \\
F7 \\
F8 \\
\end{bmatrix} = \begin{bmatrix}
\ddots \\
\ddots \\
\ddots \\
\ddots \\
\ddots \\
\ddots \\
\ddots \\
\ddots \\
\end{bmatrix} \begin{bmatrix}
f1 \\
f2 \\
f3 \\
f4 \\
f5 \\
f6 \\
f7 \\
f8 \\
\end{bmatrix}
\]

Each circle represents a complex number; x is cosine, y is sine

Fourier as Linear Transform

\[
\begin{bmatrix}
R(F(0)) & I(F(0)) & R(F(1)) & I(F(1)) & R(F(N/2)) & I(F(N/2)) \\
\cos(0) & 0 & \cos(0) & 0 & \ldots & \cos(0) & 0 \\
0 & \sin(0) & 0 & \sin(0) & \ldots & 0 & \sin(0) \\
\cos\left(\frac{2\pi \cdot 0}{N}\right) & 0 & \cos\left(\frac{2\pi \cdot 1}{N}\right) & 0 & \ldots & \cos\left(\frac{2\pi \cdot (N-1)}{N}\right) & 0 \\
0 & \sin\left(\frac{2\pi \cdot 0}{N}\right) & 0 & \sin\left(\frac{2\pi \cdot 1}{N}\right) & \ldots & 0 & \sin\left(\frac{2\pi \cdot (N-1)}{N}\right) \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\cos\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot 0}{N}\right) & 0 & \cos\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot 1}{N}\right) & 0 & \ldots & \cos\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot (N-1)}{N}\right) & 0 \\
0 & \sin\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot 0}{N}\right) & 0 & \sin\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot 1}{N}\right) & \ldots & 0 & \sin\left(\frac{2\pi \cdot \left(\frac{N}{2}\right) \cdot (N-1)}{N}\right)
\end{bmatrix}
\begin{bmatrix}
I(0) \\
I(0) \\
I(1) \\
I(1) \\
I(N-1) \\
I(N-1)
\end{bmatrix}
\]
In OpenCV – Fourier Tutorial

Discrete Fourier Transform

Goal

We'll seek answers for the following questions:

- What is a Fourier transform and why use it?
- How to do it in OpenCV?
- Usage of functions such as: `copyMakeBorder()`, `merge()`, `dft()`, `getOptimalDFTSize()`, `log()` and `normalize()`.

Source code
Helpful Images and Examples

Interpret Fourier Transforms (FFT)

Visually Interpreting FFTs
The best way to gain an intuitive understanding of what an FFT image represents is with some visual examples. The remainder of these pages assume a basic understanding of the relationship between 2D image files and their Fourier transforms. Even if you already have an appreciation of the mathematics behind FFTs of 2D image files, you may still appreciate the information on this page.

Visually Interpreting Fourier Transforms
First, here is an image of mathematically generated noise with a Gaussian distribution and its FFT:
Non-local properties

• Change one spatial pixel in an image, *every frequency value changes*

• Change one value in the frequency domain, *every spatial pixel changes.*

• Frequencies describe the image as a whole, not useful for describing part of an image
Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...

Meow!
Review: Why Study Fourier?

• Relates continuous to discrete
  – Continuous: underlying signal
  – Discrete: sampling of signal

• Tells us how much information is lost
  – Total energy in continuous frequencies above N/2

• Explains aliasing
The Nyquist Rate

• What if the frequency is above N/2?
  – You have fewer than one sample per half-cycle
  – High frequencies look like lower frequencies

Aliasing – Another View

Example by Brent Locher - www.fourier-series.com