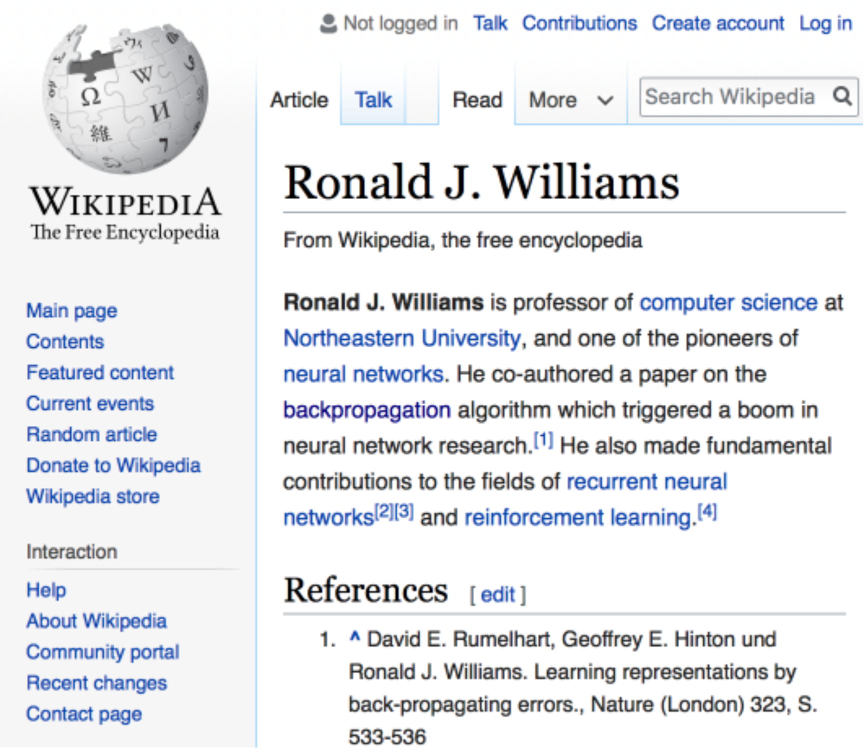
# Backpropagation

CS 510

Lecture #12

March 30th, 2020

# A Rebirth, one of several



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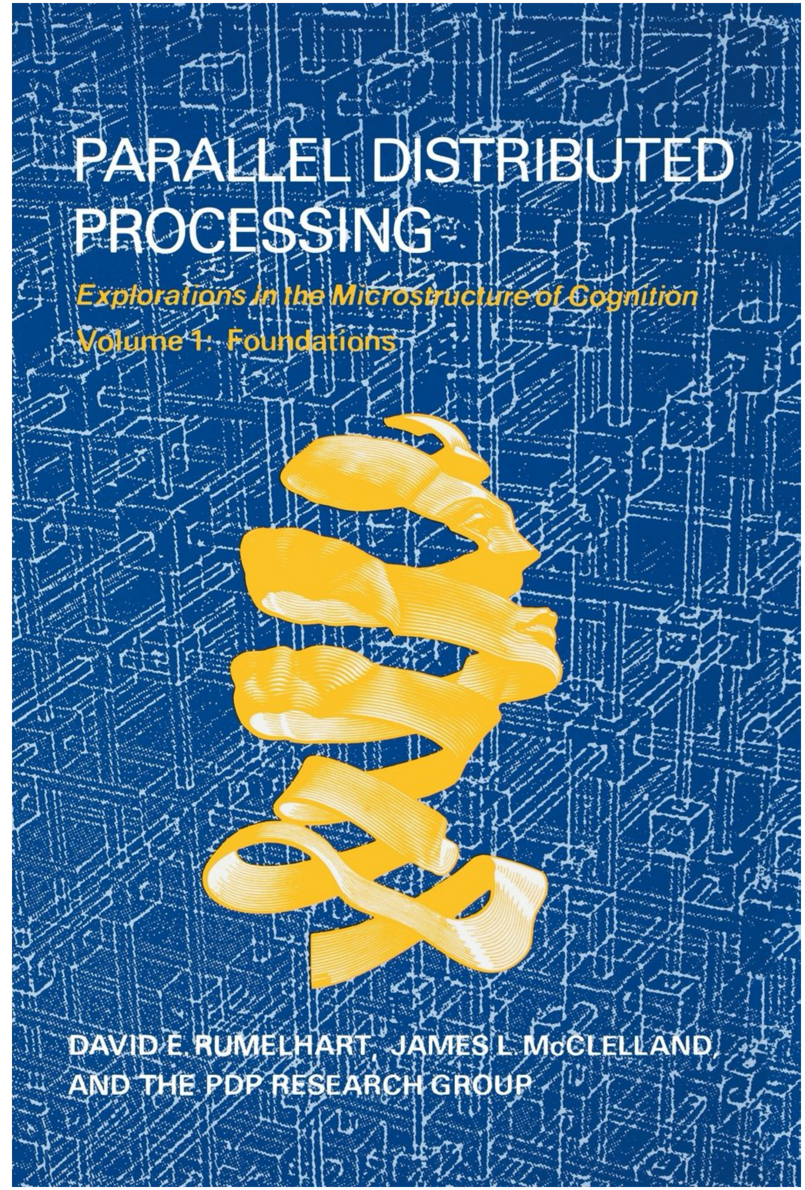
## Ronald J. Williams

From Wikipedia, the free encyclopedia

**Ronald J. Williams** is professor of computer science at Northeastern University, and one of the pioneers of neural networks. He co-authored a paper on the backpropagation algorithm which triggered a boom in neural network research.[1] He also made fundamental contributions to the fields of recurrent neural networks[2][3] and reinforcement learning.[4]

## References [ edit ]


1. ^ David E. Rumelhart, Geoffrey E. Hinton und Ronald J. Williams. Learning representations by back-propagating errors., Nature (London) 323, S. 533-536

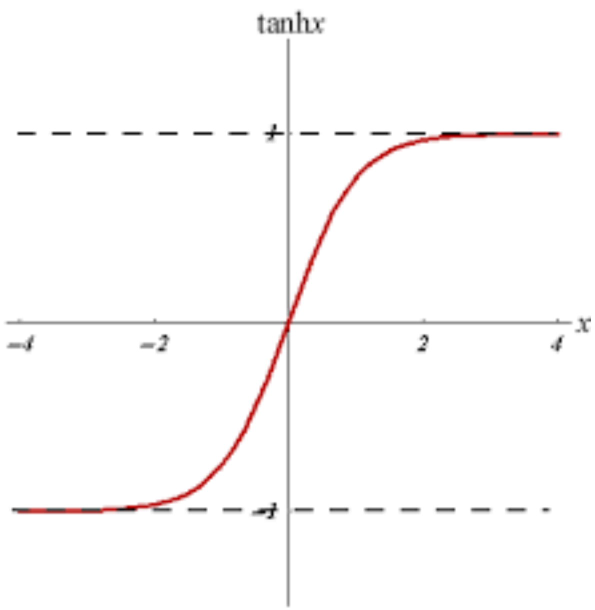



# Multi-layer Perceptrons II

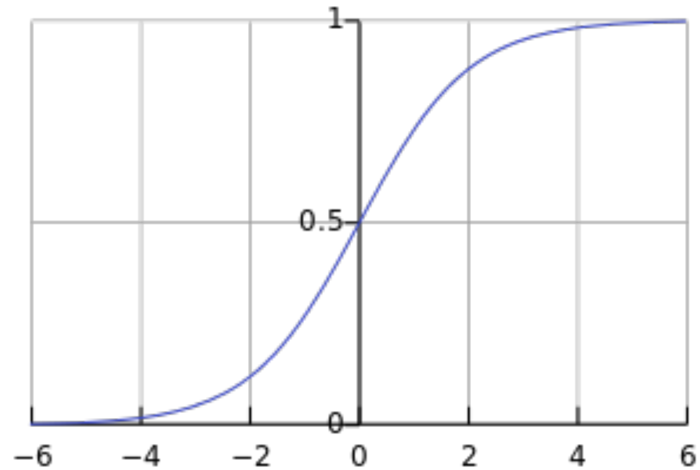
- Classic perceptrons threshold linear functions
  - $f(x) = h(w \cdot x + b)$
  - h() is a threshold-based *activation function*
  - Converts activations into decisions
- But if we want to combine perceptrons?
  - Thresholding individual perceptrons is not useful
  - Replacing h() with identity would allow us to sum linear responses
  - But a sum of linear responses is just another linear response

# Sigmoid Activation Functions

- $f(x) = s(w \cdot x + b)$




$y = tanh(x)$




$y = (1 + e^{-x})^{-1}$

# Activation Function Properties

- Activation functions <u>must</u>
  - Be non-linear
- Activation functions <u>may</u>
  - map an infinite domain to a finite range
    - Like [-1, 1] (for tanh) or [0, 1] (for logistic)
    - Keeps values from growing too large/small
    - Sometimes called "squashing"
  - Have non-zero derivatives everywhere
    - Useful for training

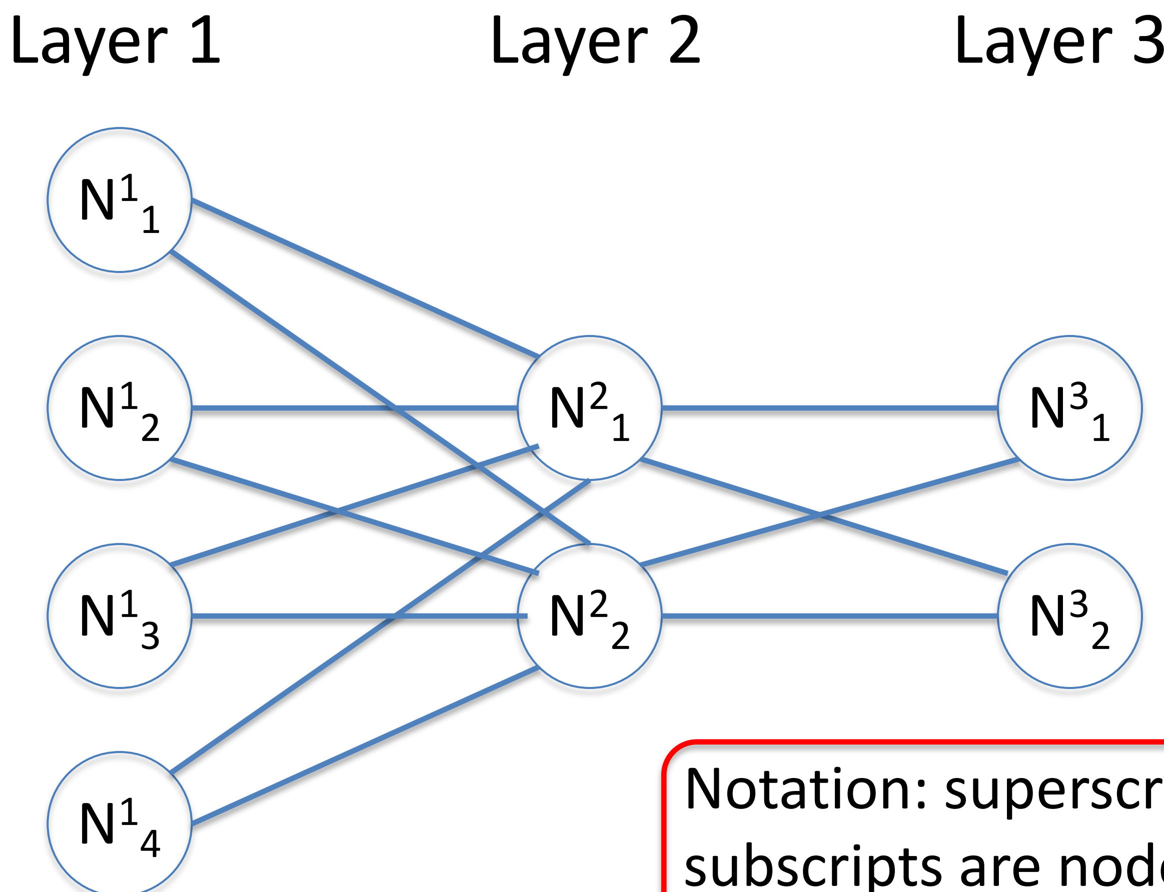
# Backpropagation

- Backpropagation is the algorithm that describes how we update weights in a network, given
  - Training samples
  - Training labels
  - A cost function
- Its used for (almost) all networks
- Network nodes may be
  - Non-linear perceptrons (the most common)
  - Convolutional units
  - Pooling units
  - Batch normalization units
  - …

# Goals For Today

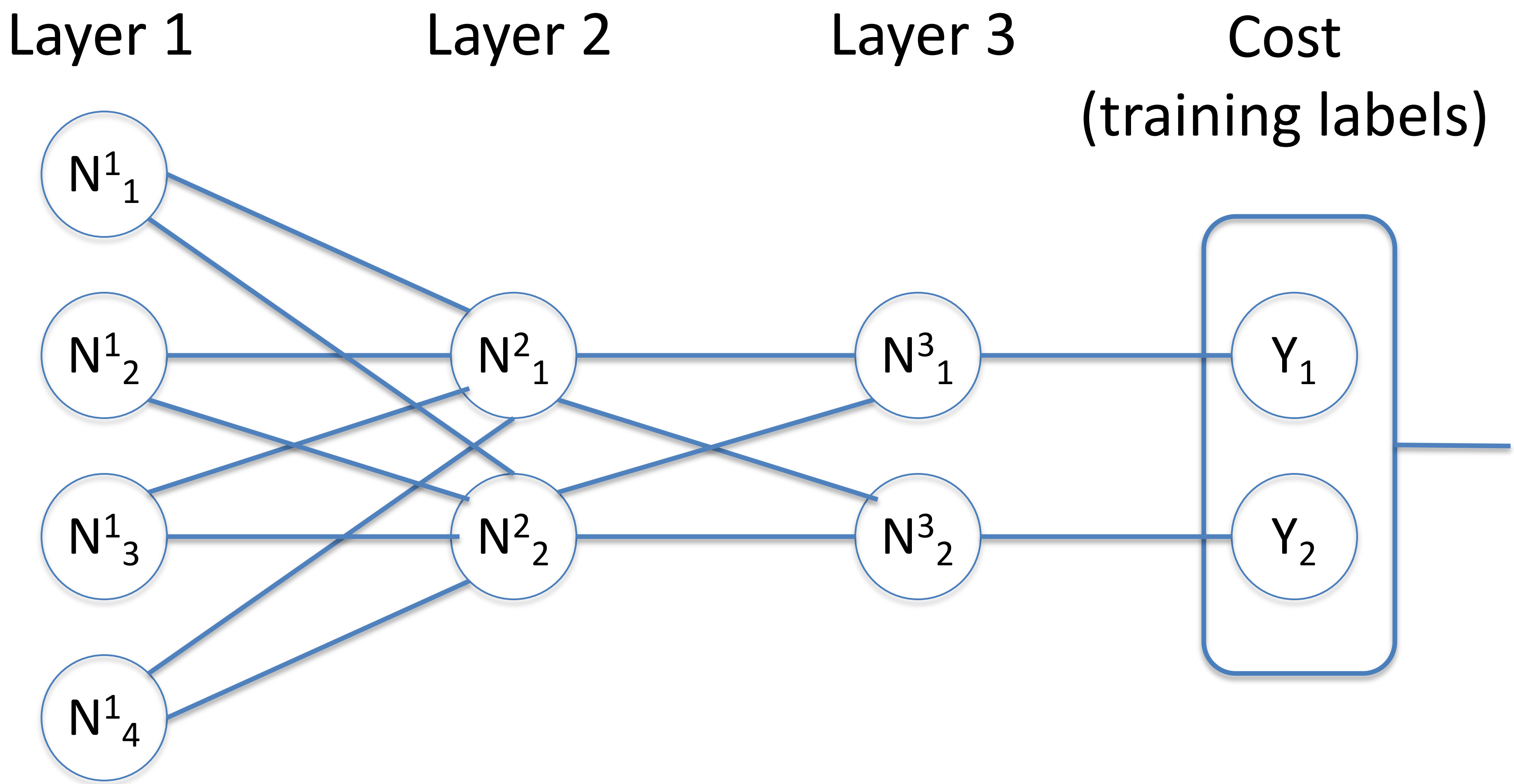
- Walk you through the math of backpropagation
  - Complicated, but just calculus
  - Almost universal : modifiable for different node types (see previous slide)
- Today's derivation assumes multi-layer perceptrons
  - $z(x) = wx + b$
  - $a(x) = h\big(z(x)\big) = h(wx + b)$
- Remember the chain rule from calculus:
  - $f(x) = g\big(h(x)\big) \rightarrow f'(x) = g'\big(h(x)\big)h'(x)$

# Simple Neural Network




Notation: superscripts are layers, subscripts are node numbers

# Setup for Training: Cost



# Cost Functions

- Cost functions measure the gap between the network output and the ideal output
- Two necessary properties

*Allows us to optimize per sample*

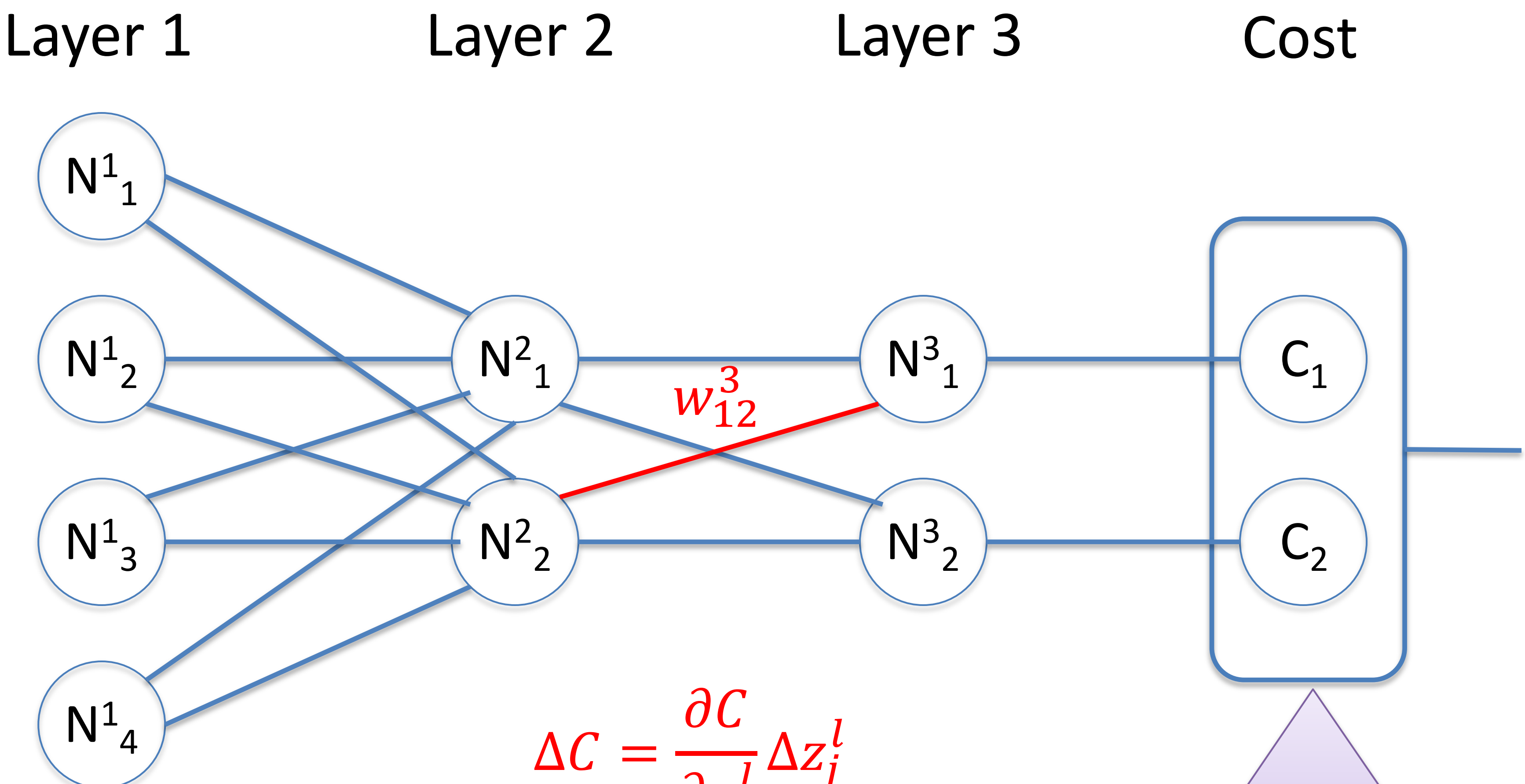
  1. An average over samples: $C = \frac{1}{n}\sum_x C_x$
  2. Function of output activations: $C = C(a_l)$

- Example: mean squared error

$$C = \frac{1}{2n}\sum_x \|y(x) - a^L(x)\|^2$$

*Allows us to initialize the partial derivative computations*

# δs : local derivatives as error measures




$$\Delta C = \frac{\partial C}{\partial z^l_j} \Delta z^l_j$$

Subtle change from previous diagram, now showing cost C not training label.

# Partial derivatives as error measures

- Imagine you want to change the output $z^l_{j,}$ by $\Delta z^l_j$
- Then $\Delta C = \frac{\partial C}{\partial z_j^l} \Delta z_j^l$
- If $\left|\frac{\partial C}{\partial z_j^l}\right|$ is large, then C becomes smaller by giving $\Delta z_j^l$ the opposite sign
- But if $\left|\frac{\partial C}{\partial z_j^l}\right|$ is near zero, then $\Delta z_j^l$ doesn't matter.
  - $\frac{\partial C}{\partial z_j^l}$ is already optimal!
  - $\delta_j^l \equiv \frac{\partial C}{\partial z_j^l}$

# Recap - where are we?

- We can optimize on a per-sample basis
  - Because the cost function is an average
- Minimizing the δs optimizes the net
  - The δs depend on the data samples
- But how do we minimize the δs?

- We will assume that nodes have non-linear functions, so $a_j^l = h(z_j^l)$

# Output Layer

- $\delta_j^L = \frac{\partial C}{\partial a_j^L} h'\left(z_j^L\right)$ by the chain rule
- $\frac{\partial C}{\partial a_j^L}$ is the partial derivative of C with respect the activation of output unit j
  - If C is LMS (slide #6)
    - $\frac{\partial C}{\partial a_j^L} = a_j^L(x) - y(x)$
    - The difference between the output & desired output

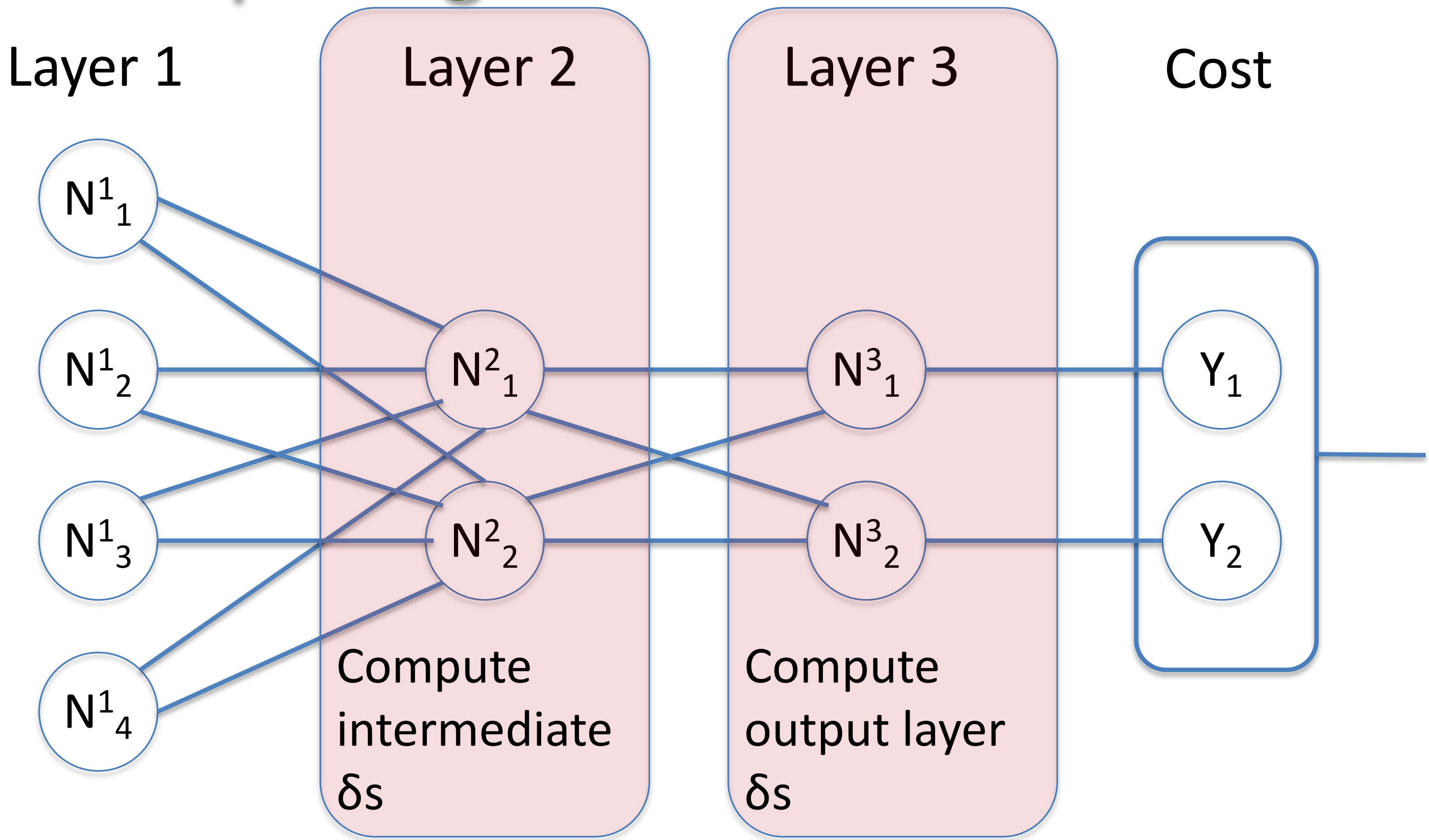
# Output Layer (cont.)

- $h'\left(z_j^L\right)$ is the derivative of the non-linear transfer function at $z_j^L$
- If $h(x) = tanh(x)$, $\sigma'(x) = 1 - tanh^2(x)$
- If $h(x) = (1 + e^{-x})^{-1}$,
$$\sigma'(x) = \sigma(x)\big(1 - \sigma(x)\big)$$
- $\delta_j^L = \left(a_j^L(x) - y(x)\right)\left(1 - tanh^2\left(z_j^L(x)\right)\right)$ <u>or</u>
- $\delta_j^L = \left(a_j^L(x) - y(x)\right)\left(a_j^L(x)\left(1 - a_j^L(x)\right)\right)$

# $\delta^{L}$ given $\delta^{L+1}$

- $\delta_j^l = \sigma'\left(z_j^l\right) \sum_k w_{kj}^{l+1} \delta_k^{l+1}$
- σ' is computed as on previous slide
- The RHS is just the sum of the impacts
- This is where *backpropagation* comes from
  - Calculate δs for output layer
  - Then recursively compute δs for previous layers

# Computing δs…



# So…

- Given an input x and output y:
  - We can compute $\delta^l_j$ for every node j at every level l
  - Minimizing the δs will optimize the network
    - Relative to this sample
  - So we need to adjust the weights $w_i$ and b to reduce the δs
    - But just a little for each input/output pair
    - So we can optimize across all samples

# Adjusting b

- Remember that $\delta_j^l \equiv \frac{\partial C}{\partial z_j^l}$ (slide #8)
- And that $z_j^l = w_j^l x + b$
- So $\frac{\partial C}{\partial b_j^l} = \delta_j^l$
- So $b_j^l \leftarrow (1 - \alpha) b_j^l - \alpha \delta_j^l$
  - Where α is a learning rate
  - Regulates how much you react to each sample

# Adjusting w's

- $\frac{\partial C}{\partial w_{jk}^{l}} = a_k^{l-1}\delta_j^l$
- So $w_{jk}^{l} \leftarrow (1-\alpha)w_{jk}^{l} - \alpha a_k^{l-1}\delta_j^l$
  - Where α is the same learning rate as before
  - We are collectively minimizing the deltas by heading downhill in the k+1 dimensional space defined by w & b

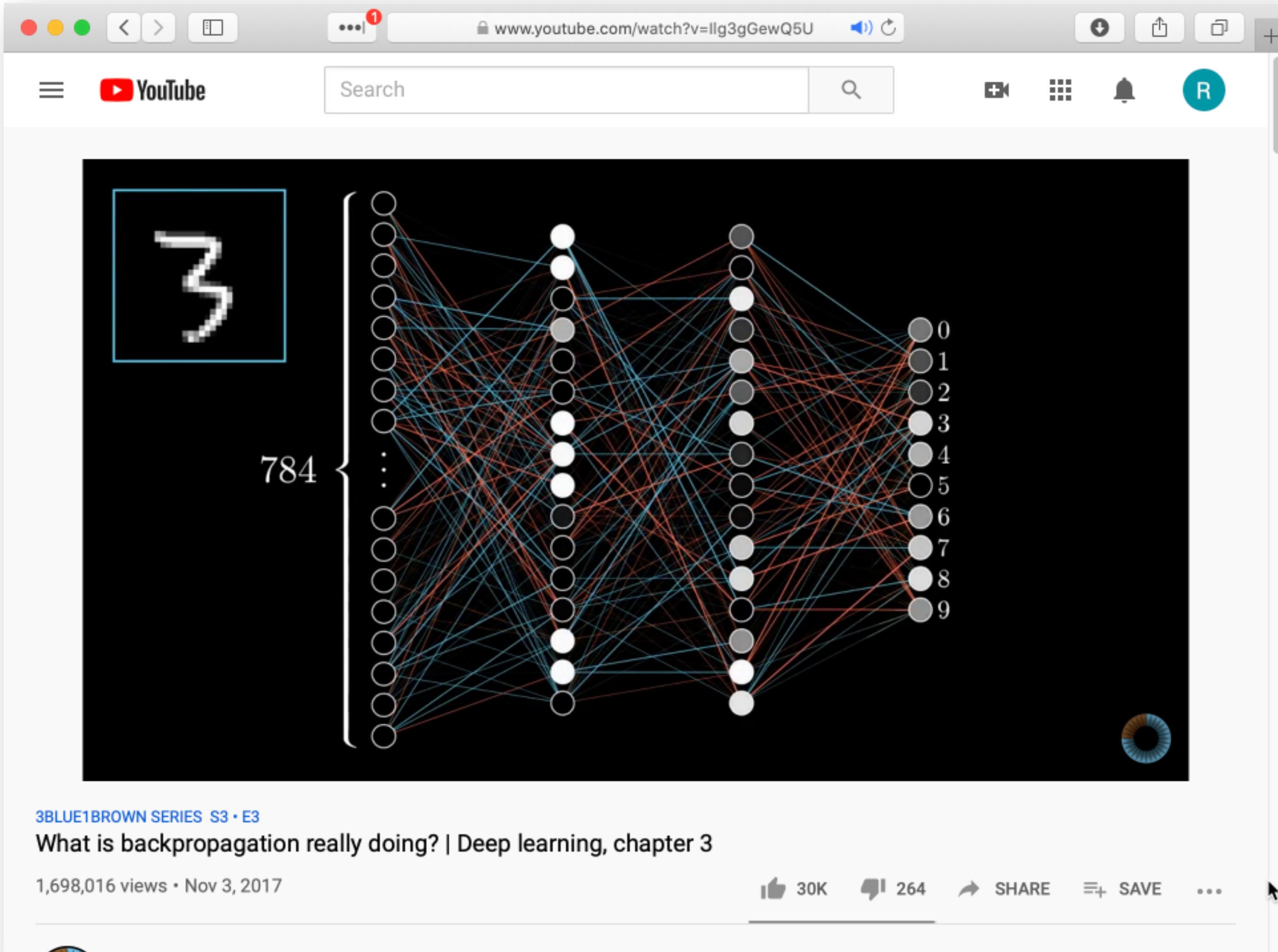
# Backpropagation (redux)

- Backpropagation updates weights in a network, given
  - Training samples
  - Training labels
  - A cost function
- Network nodes may be
  - Non-linear perceptrons (the most common)
  - Convolutional units
  - Pooling units
  - Batch normalization units
  - …

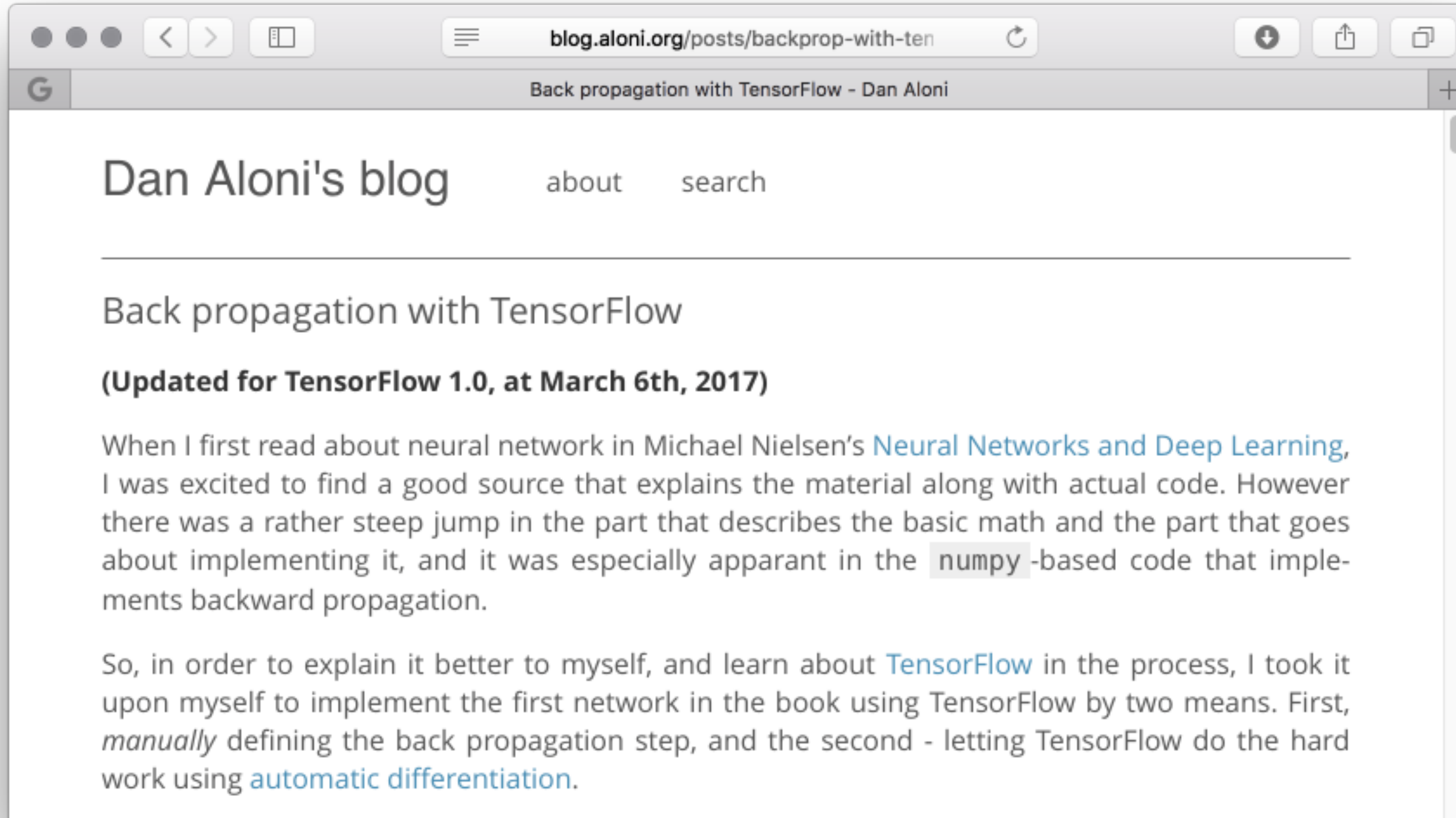
# Step Back!  Other Resources

- Modulo some notation ambiguity the previous formula-based presentation if fine, but for some of us unsatisfying
- It is best to approach the task of understanding backpropogation simultaneously from three angles.
    1. Mathematical formulas (just finished)
    2. Develop an internal visualization
    3. Running code

# May I Recommend



# And Also


blog.aloni.org/posts/backprop-with-ten
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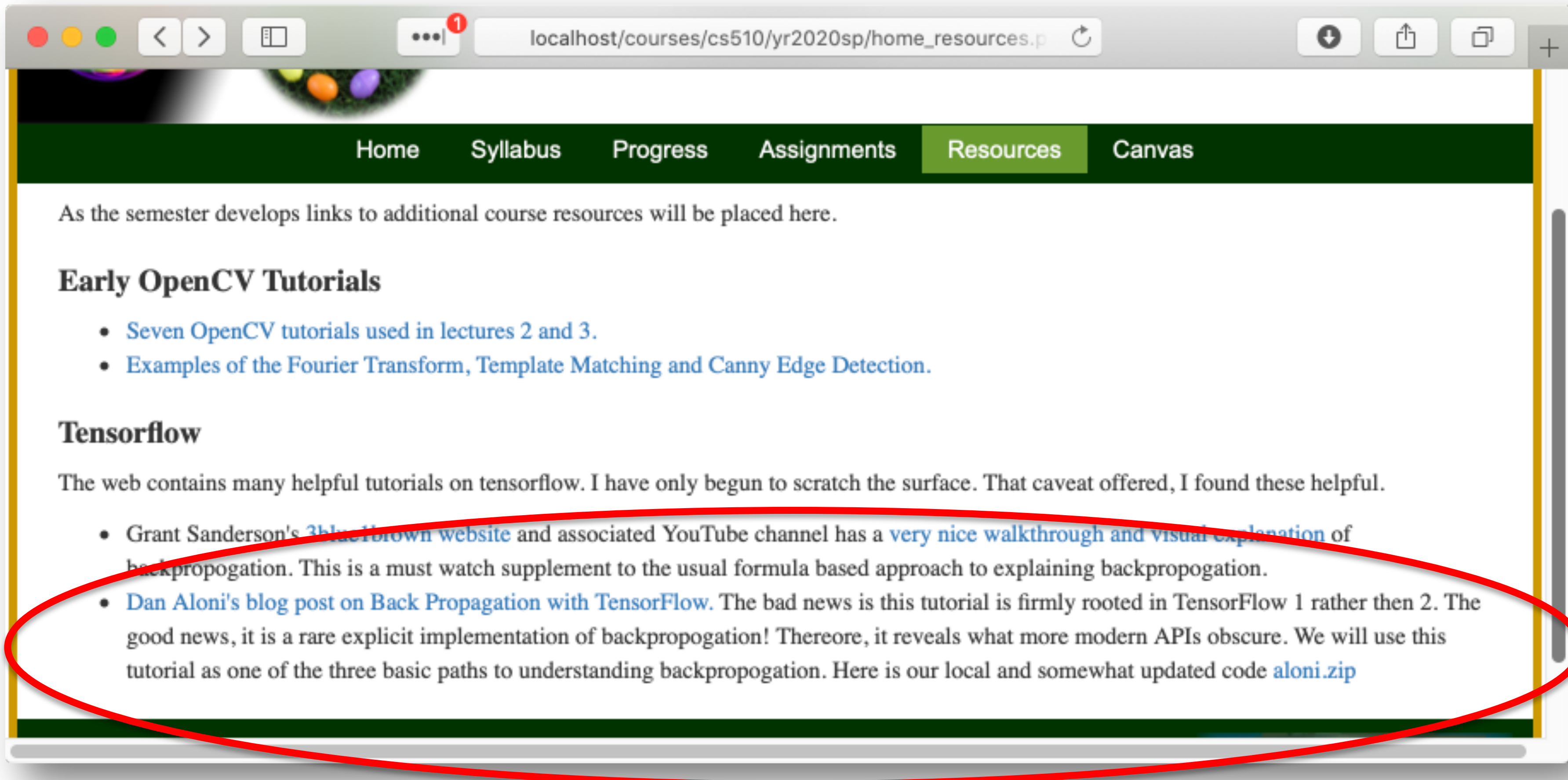
Back propagation with TensorFlow
(Updated for TensorFlow 1.0, at March 6th, 2017)
When I first read about neural network in Michael Nielsen's Neural Networks and Deep Learning, I was excited to find a good source that explains the material along with actual code. However there was a rather steep jump in the part that describes the basic math and the part that goes about implementing it, and it was especially apparant in the numpy -based code that implements backward propagation.
So, in order to explain it better to myself, and learn about TensorFlow in the process, I took it upon myself to implement the first network in the book using TensorFlow by two means. First, manually defining the back propagation step, and the second - letting TensorFlow do the hard work using automatic differentiation.

# Please Run and Play With …




As we will discuss in lecture today, I expect everyone to setup a TF 1.14 environment and play with this code.