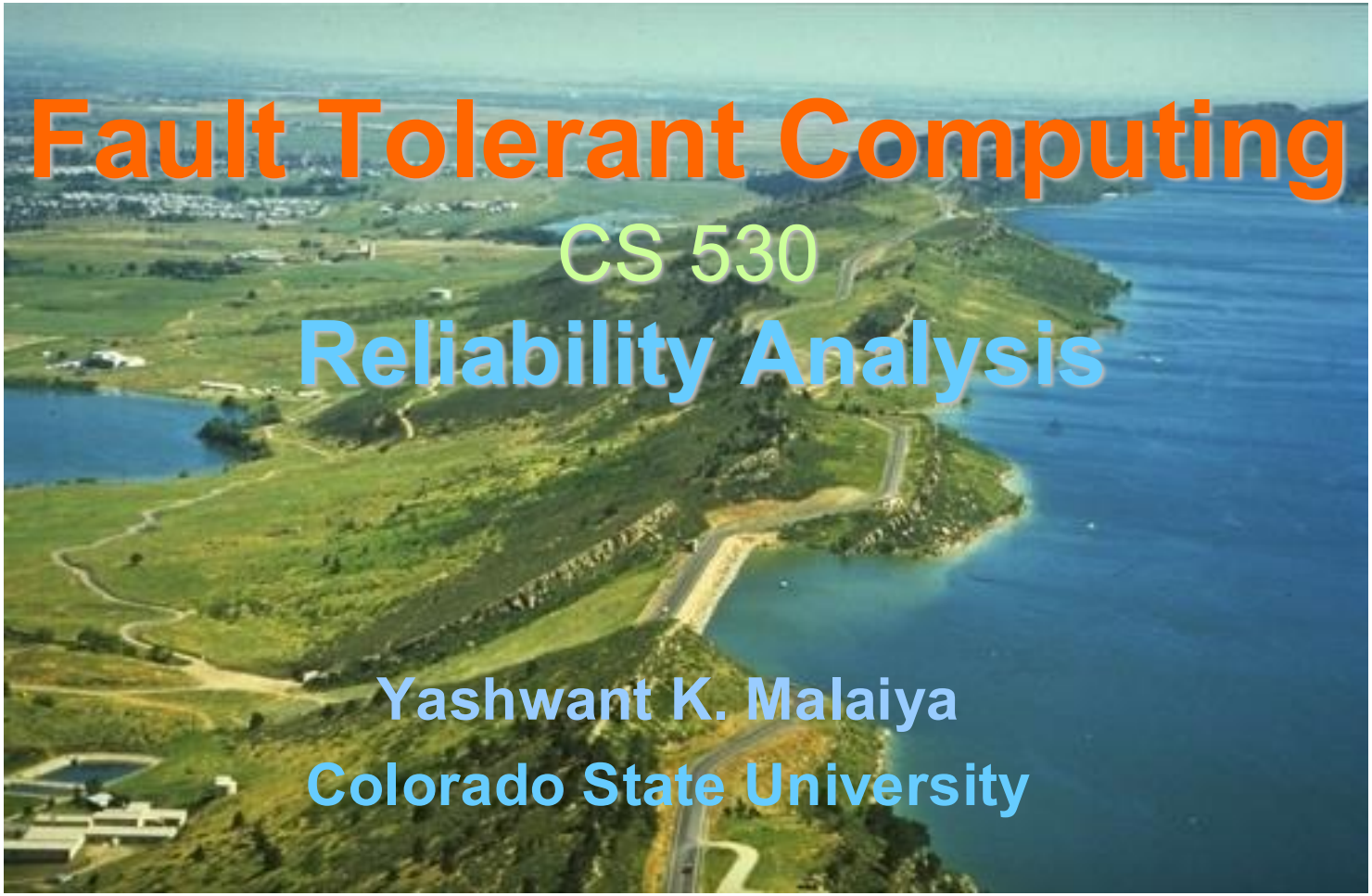

Fault Tolerant Computing

CS 530

Reliability Analysis

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Colorado State University

Reliability Analysis: Outline

Reliability measures:

- **Reliability, availability, Transaction Reliability,**
- **MTTF and R(t), MTBF**

Basic Cases

- **Single unit with permanent failure, failure rate**
- **Single unit with temporary failures**

Combinatorial Reliability: Block Diagrams

- **Serial, parallel. K-out-of-n systems**
- **Imperfect coverage**

Redundancy

- **TMR, spares**
- **Generalized**

Reliability Analysis

- **Permanent faults**
  - **The unit will eventually fail. Thus reliability "decays".**
- **Temporary faults**
  - **Faults come and go. Often Steady state characterization is possible.**
  - **Permanent faults subject to repair are modeled as temporary faults.**
- **Design faults**
  - **Reliability growth occurs during testing & debugging. We will study this under "Software Reliability" later.**

Why Mathematical Analysis?

- You can determine reliability by constructing a large number of copies of the target system, and collecting failure data. However, that would be infeasible except for special cases.
- Thus we need to be able to determine the reliability before a system is built, by using the information we have about the components and the proposed architecture.

Basic Reliability Measures

- **Reliability:** durational (default)

  R(t)=P{correct operation in duration (0,t)}

  - This is the default definition of reliability.
- **Availability:** instantaneous

  A(t)= P{correct operation at instant t)}

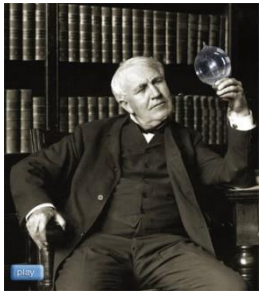
  - Applied in presence of temporary failures
  - A steady-state value is the expected value over a range of time.
- **Transaction Reliability:** single transaction

  R_t=P{a transaction is performed correctly}
- The term “Reliability” is sometimes used with a non-standard meaning.

Mean time to …

- **Mean Time to Failure (MTTF):** expected time the unit will work without a failure.
- **Mean time between failures (MTBF):** expected time between two successive failures.
  - Applicable when faults are temporary.
  - The time between two successive failures includes repair time and then the time to next failure.
  - Approximately equal to
- **Mean time to repair (MTTR):** expected time during which the unit is non-operational.

Mean time to …



Average Rated Life for Various Types of Bulbs

Type	Hours	Power
Incandescent	750-2,000	100%
Compact Fluorescent CFL		25%
Plug-in	10,000-20,000	
Screw-based	8,000-10,000	
Halogen	2,000-4,000	
LED	10,000-50,000	15%

The Great Lightbulb Conspiracy: The Phoebus cartel engineered a shorter-lived lightbulb and gave birth to planned obsolescence IEEE Spectrum

Mean Time to Failure (MTTF)

- There is a very useful general relation between MTTF and R(t). Here T is time to failure, which is a random variable.

$$MTTF = E(T) = \int_0^\infty t\, f(t)dt$$

$$= -\int_0^\infty t\frac{dR(t)}{dt}dt$$

$$= [-tR(t)]_0^\infty + \int_0^\infty R(t)dt$$

$$Thus\ MTTF = \int_0^\infty R(t)dt$$

Worth Remembering!

Note:

$$R(t) = 1 - P\{failure\ in\ (0,t)\}$$
$$= 1 - P\{0 \le T \le t\}$$
$$= 1 - F(t)$$
$$\therefore \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$
$$or\ f(t) = -\frac{dR(t)}{dt}$$

Note:

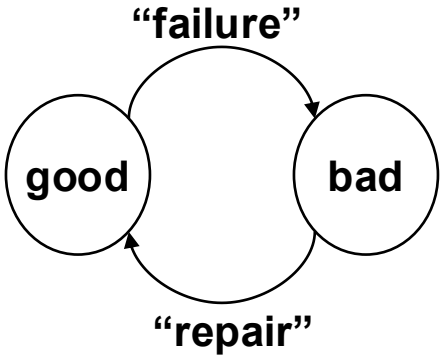
$xe^{-x} \to 0\ as\ x \to \infty$

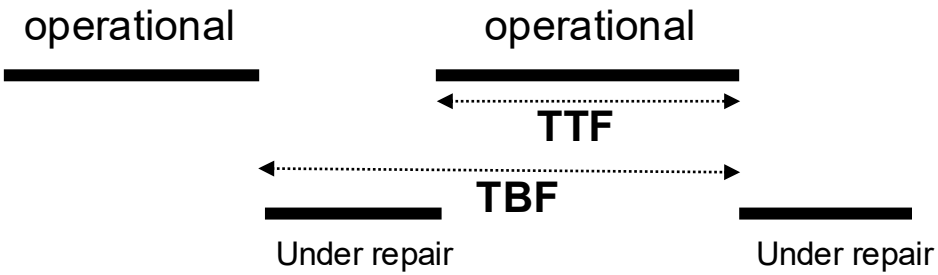
and R(t) is generally of the form e^{-at}

Thus $tR(t) \to 0\ as\ t \to \infty$.

Failures with Repair

- **Time between failures: time to repair + time to next failure**





- **MTBF = MTTF + MTTR**
  - MTBF, MTTF are same same when MTTR ≈0
- **Steady state availability = MTTF / (MTTF+MTTR)**

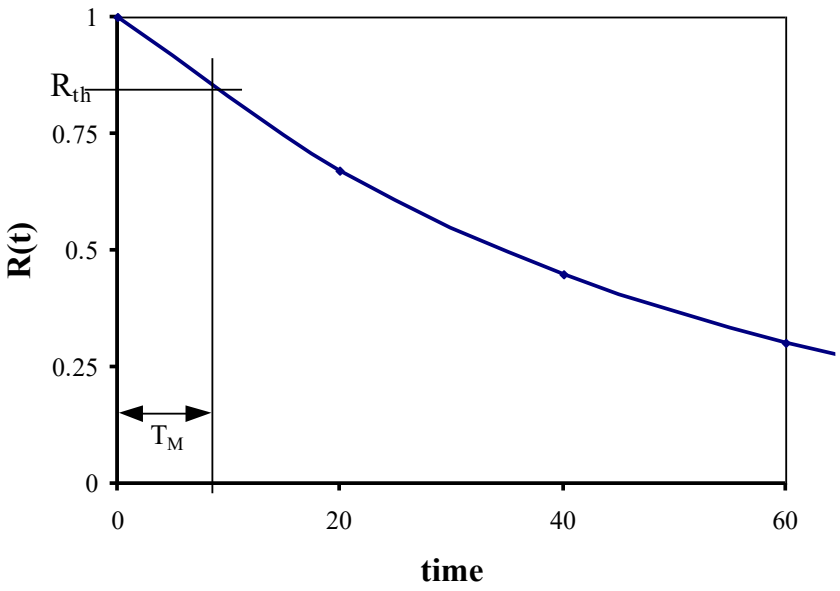
Cloud Observed uptime 2025

Cloud Provider	Observed Uptime	Mean time to recovery
Amazon Web Services (AWS)	99.95%	2.8 hours
Microsoft Azure	99.97%	4.2 hours
Google Cloud Platform (GCP)	99.98%	1.8 hours

- 99.97% availability means total downtime 2.63 hours per year
- Redundancy can increase availability. Can be across multiple availability zones or even regions.
- SLA (*Service Level Agreement*): If availability falls below 99.9%, customers receive service credits.
- Durability: Will my data ever be permanently lost? Est: 99.999999999%

Mission Time (High-Reliability Systems)

- Reliability throughout the missic must remain above a threshold reliability R_{th}.
- Mission time T_M: defined as the duration in which $R(t) \geq R_{th}$.
- R_{th} may be chosen to be perhaps 0.95.
- Mission time is a strict measure, used only for very high reliability missions.



Two Basic cases

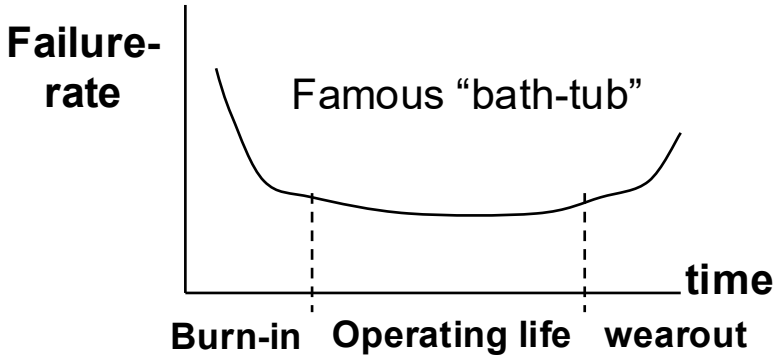
- We next consider two very important basic cases that serve as the basis for time-dependent analysis.

1. Single unit subject to permanent failure
   - We will assume a constant failure rate to evaluate reliability and MTTF.
2. Single unit with temporary failures
   - System has two states Good and Bad, and transitions among them are defined by transition rates.

- Both of these are example of Markov processes.

Constant Failure Rate Assumption

- **We will always assume a constant failure rate.**
  - **It keeps analysis simple.**
  - **During operating life, the failure rate is approximately constant.**
- **The Bath-Tub curve:**
  - **In the beginning the failure rate is high because the weaker devices fail due to "infant mortality". Near the end the failure rate is again high due to "aging" or wear-out of devices.**



Policies

https://www.cs.colostate.edu/~cs530dl/

- Use of electronic devices (phones, pads or laptops) is not permitted in during the class.
- No collaboration of any type is permitted among the students in homework assignments and quizzes.

Term Project

We need technical diversity.

Project teams:

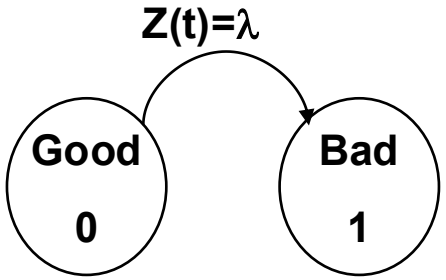
- **Two person teams**
- **Topic list: same as QuickResearch. For additional topics, contact the professor**
- **Claim topic in Project Intent Discussion group: title and one sentence description by 3/4/26.**
- **FTC must be focus and not peripheral.**

Course notes

- **Problem Set 1 (PSA1) available 2/25/26, due 3/4/26**
- **Midterm 3/12/26 Thurs**
- **Project proposal due 3/11/26**

Basic Cases: Single Unit with Permanent Failure

- **Failure rate is the probability of failure/unit time**
- **Assumption: constant failure-rate λ**



The state transition diagram & the differential equation represent What we call Markov Modeling.

$$\frac{dp_0(t)}{dt} = -\lambda\ p_0(t) \quad \text{since the rate of leaving state 0 depends on probabilit y of being in state 0}$$

$$p_0(0) = 1 \quad \text{initial condition}$$

Single Unit with Permanent Failure (2)

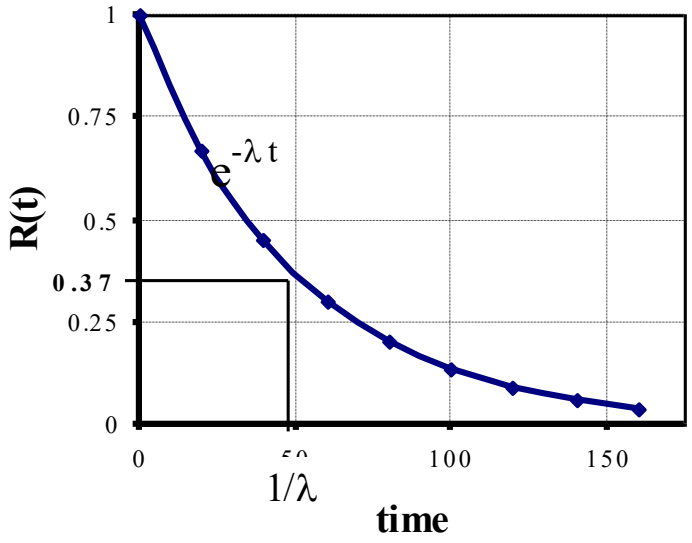
$$\frac{dp_0(t)}{dt} = -\lambda\ p_0(t)$$

$$p_0(0) = 1$$

$$Solution: p_0(t) = e^{-\lambda t}$$

$$Since\ R(t) = p_0(t)$$

$$R(t) = e^{-\lambda t}$$



"The Exponentia l reliabilit y law "

$$At \quad t = \frac{1}{\lambda},\ R(t) = e^{-1} = 0.368$$

Single Unit: Permanent Failure (3)

$$R(t) = e^{-\lambda t}$$

$A(t)$ *is same as* $R(t)$ *in this case.*

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t}dt$$

$$= [-\frac{e^{-\lambda t}}{\lambda}]_0^\infty$$

$$= \frac{1}{\lambda}$$

- **Ex 1:** a unit has MTTF =30,000 hrs. Find failure rate.

  $\lambda = 1/30{,}000 = 3.3 \times 10^{-5}$/hr
- **Ex 2:** Compute mission time T_M if $R_{th} = 0.95$.

  $e^{-\lambda T_M} = 0.95 \quad T_M = -\ln(0.95)/\lambda$

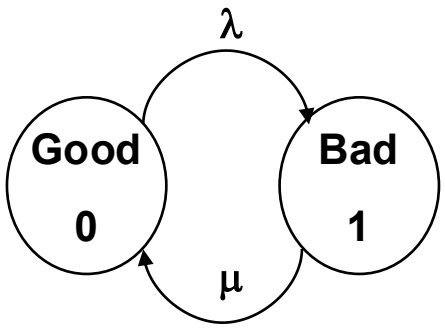
  $\approx 0.051/\lambda$
- **Ex 3:** Assume $\lambda = 3.33 \times 10^{-5}$, and $R_{th} = 0.95$ find T_M.

  Ans: T_M = 1538.8 hrs

  (compare with MTTF =30,000)

Single Unit: Temporary Failures(1)

- **Temporary: intermittent, transient, permanent with repair**



$$\frac{dp_0(t)}{dt} = -\lambda\ p_0(t) + \mu\, p_1(t)$$

$$\frac{dp_1(t)}{dt} = +\lambda\ p_0(t) - \mu\, p_1(t)$$

Note state diagram & Differential equations for Markov modeling

can be solved by laplace transform etc.

$$p_0(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$$

Similarly we can get an expression for $p_1(t)$, however it is not needed since $p_1(t) = 1 - p_0(t)$.

Su, Koren, Malaiya, "A Continuous-Parameter Markov Model and Detection Procedures for Intermittent Faults," , IEEE Tran on Computers, vol.C-27, no.6, pp.567,570, June 1978

Single Unit: Temporary Failures(2)

- $p_0(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1-e^{-(\lambda+\mu)t})$
- Availabili ty $A(t) = p_0(t)$

Thus $A(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1-e^{-(\lambda+\mu)t})$

- Note that steady – state probabilit ies exist :

$$t \to \infty, p_0(t) = \frac{\mu}{\lambda+\mu} \qquad p_1(t) = \frac{\lambda}{\lambda+\mu}$$

- Steady - state availabili ty is $\frac{\mu}{\lambda+\mu}$

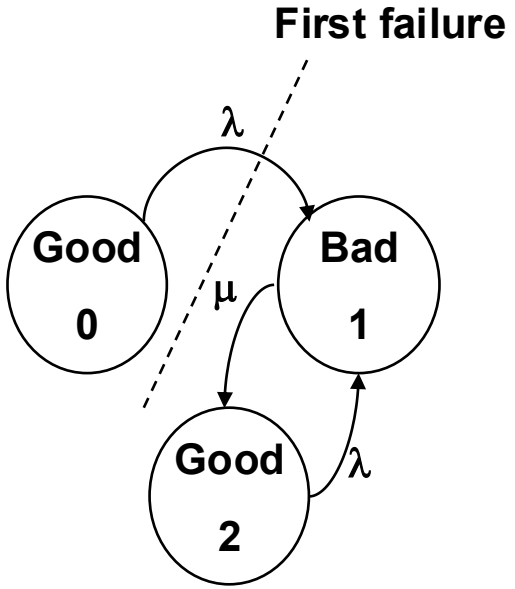
Single Unit: Temporary Failures(3)

- Reliability (durational)

$$R(t) = P\{\text{no failures in } (0,t)\}$$
$$= P\{\text{in Good 0 at } t\}$$
$$= e^{-\lambda t}$$

same as permanent failure

- Thus $\text{MTTF} = \frac{1}{\lambda}$
- Mission time : also same



Y. K. Malaiya, S. Y. H. Su: Reliability Measure of Hardware Redundancy Fault-Tolerant Digital Systems with Intermittent Faults. IEEE Trans. Computers 30(8): 600-604

Note that when we say **no failures in (0,t),** even a brief failure is a failure. Thus R(t) may be too strict a measure when brief failures may be acceptable.

Combinatorial Reliability

This is a part of classic reliability theory.

Objective is: Given a

- systems structure in terms of its units
- reliability attributes of the units
- some simplifying assumptions

- We need to **evaluate the overall reliability** measure.

There are **two extreme cases** we will examine first:

- Series configuration
- Parallel configuration
- Other cases involve combinations and other configurations.

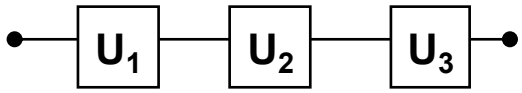
- Note that conceptual modeling is applicable to R(t), A(t), $R_t(t)$. A system is either good or bad.

Series configuration

Series configuration: all units are essential, each unit has a different functionality. System fails if any one of them fails.

- **Assumption:** statistically independent failures in units.

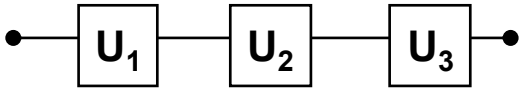
$$R_S = P\{U_1\ good \cap U_2\ good \cap U_3\ good\}$$
$$= P\{U_1\ g\}P\{U_2\ g\}P\{U_3\ g\}$$
$$= R_1R_2R_3$$



$$\text{In general } R_S = \prod_{i=1}^{n} R_i$$

Series configuration

The reliability block diagrams like this are only conceptual, not physical.



If $R_i(t) = e^{-\lambda_i t}$

then $R_s(t) = \Pi e^{-\lambda_i t} = e^{-[\lambda_1 + \lambda_2 + \cdots + \lambda_n]t}$

i.e. system failure rate is the sum of individual failure rates :

$\lambda_S = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

This gives us a nice way to estimate the overall failure rate, when all the individual units are essential. This is the basis of the approach used in the popular "Military Handbook" MIL-HDBK-217 approach for estimating the failure rates for different systems.

The failure rates of individual units are estimated using empirical formulas. For example the failure rate of a VLSI chip is related to its complexity etc.

"A chain is as strong as it's weakest link"

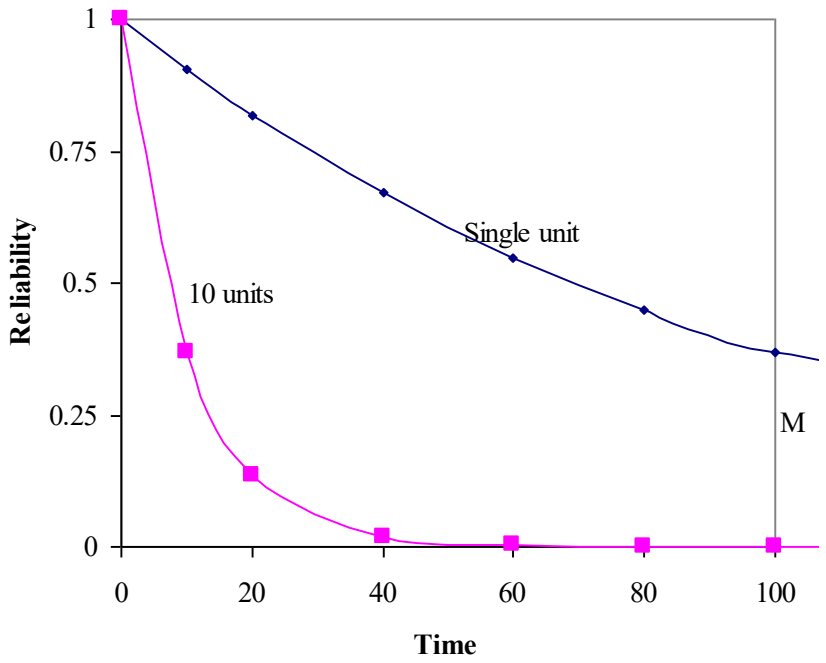
Do you agree?

Let us see for a 4-unit series system

- Assume $R_1=R_2=R_3=0.95$, $R_4=0.75$
  - R_S= 0.95x0.95x0.95x0.75 =0.643
- Thus a chain is slightly weaker than its weakest link!

The plot gives reliability of a 10-unit system vs a single system. Each of the 10 units are identical.

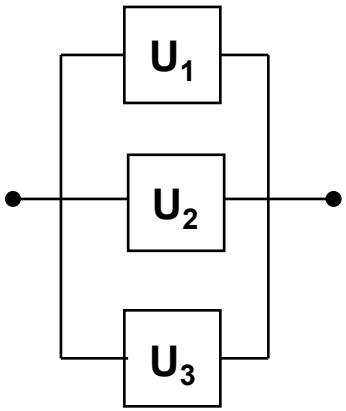
- More units, less reliability.



Combinatorial: Series

Combinatorial: Parallel

- **Parallel configuration:** System is good when least one of the several replicated units is good. A parallel configuration represents an *ideal* redundant system, ignoring any overhead.



Identical functionality, implementation may be different.

$$R_s = 1 - P\{all\ units\ \ bad\}$$
$$= 1 - P\{U_1\ bad \cap U_2\ bad \cap U_3\ bad\}$$
$$= 1 - P\{U_1\ b.\}P\{U_2\ b.\}P\{U_3\ b.\}$$
$$= 1 - (1-R_1)(1-R_2)(1-R_3)$$

$$\text{In general } R_s = 1 - \prod_{i=1}^{n}(1-R_i)$$

Example

1-(1-0.8)(1-0.8)

=1-0.04 = 0.96

$$i.e. \quad \overline{R}_s = \prod_{i=1}^{n} \overline{R}_i$$

Where $\overline{R}$ represents 1-R, i.e. "*unreliability*"

Parallel Configuration: Example

Problem : Need system reliability $R_s = 1 - \epsilon$

How many parallel units are needed

if $R_1 = R_2 = \cdots = R_m, R_m < R_s$?

Sometimes it is more convenient to talk in terms of "*unreliability*"

Solution : $1 - R_s = (1 - R_m)^x$

$\epsilon = (1 - R_m)^x$

$$x = \frac{\ln \epsilon}{\ln(1 - R_m)}$$

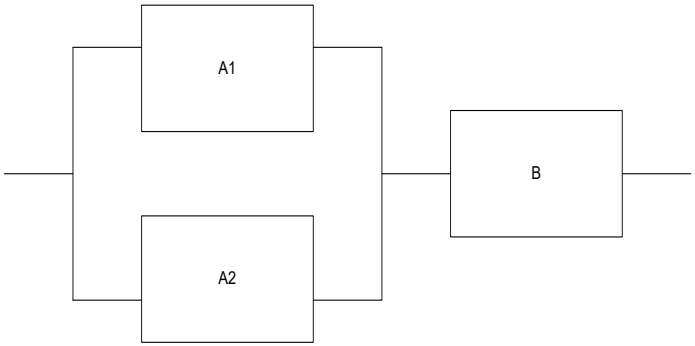
Assume $R_s = 0.9999$ ($\epsilon = 0.0001$),

$R_m = 0.9$

gives $x = 4$.

Remember, we're consider an *ideal* system

An Example Problem Pt.1



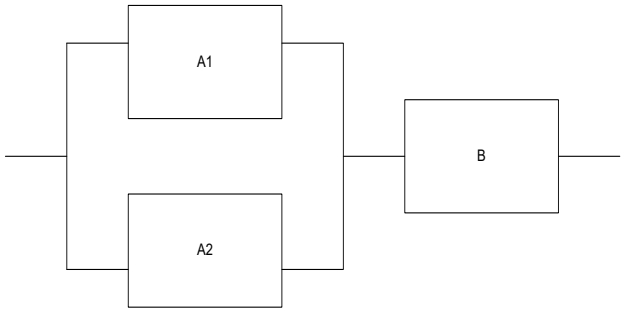
The sub-units A1 and A2 are functionally identical. System is working if any one of them are working. Unit B has some other function.

Assume independent failures.

Find an expression for R(t) and MTTF.

- **R(t) = P{ (A1 is good ∪ A2 is good) ∩ B is good}**
  = [1- P{A1 is bad}P{A2 is bad}] x P{B is good}

An Example Problem Pt.2



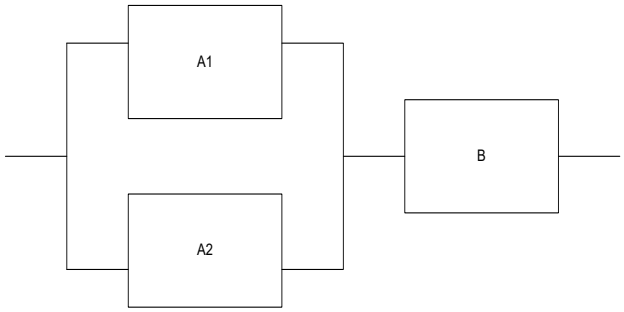
The failure rate for sub-units A1 and A2 is λ_A, for sub-unit B the failure rate is λ_B, You can assume independence of failures for sub-units.

Find an expression for R(t) and MTTF.

- **R(t) = P{ (A1 is good ∪ A2 is good) ∩ B is good}**
  = [1- P{A1 is bad}P{A2 is bad}] ∩ P{B is good}
  = [$1-(1-e^{-\lambda_A t})^2$**]**$e^{-\lambda_B t}$ **=** **[2**$e^{-\lambda_A t}-e^{-2\lambda_A t}$**]**$e^{-\lambda_B t}$
  = [2$-e^{-\lambda_A t}$**]** $e^{-(\lambda_A+\lambda_B)t}$

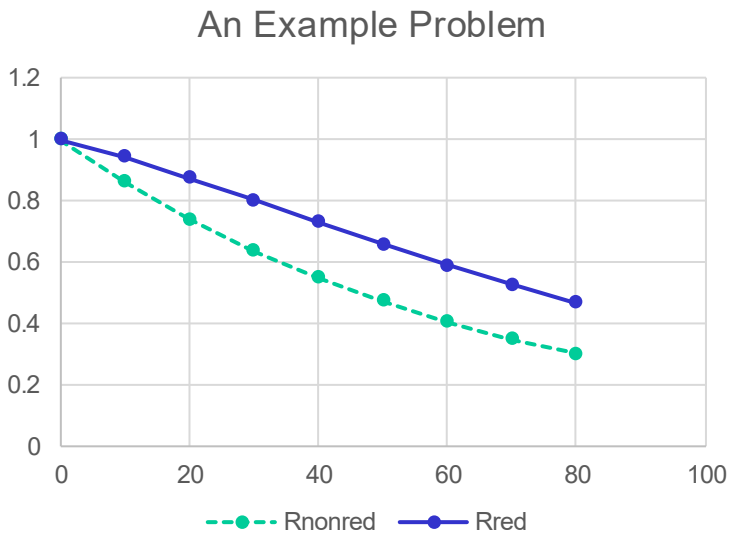
- $MTTF = \int_0^\infty R(t)dt = \int_0^\infty [2-e^{-\lambda_A t}]\, e^{-(\lambda_A+\lambda_B)t}dt =$
  $2\int_0^\infty e^{-(\lambda_A+\lambda_B)t}dt - \int_0^\infty e^{-(2\lambda_A+\lambda_B)t}dt = \frac{2}{\lambda_A+\lambda_B} - \frac{1}{2\lambda_A+\lambda_B}$

An Example Problem Pt.3



λ_A = 0.01 /h $\lambda_B = 0.005/h$

$MTTF_{nonred}$ = 66.66 h

$MTTF_{red}$ = 93.33 h

- **R(t) = [2**$-e^{-\lambda_A t}$ **]** $e^{-(\lambda_A+\lambda_B)t}$
- $MTTF = \frac{2}{\lambda_A+\lambda_B} - \frac{1}{2\lambda_A+\lambda_B}$